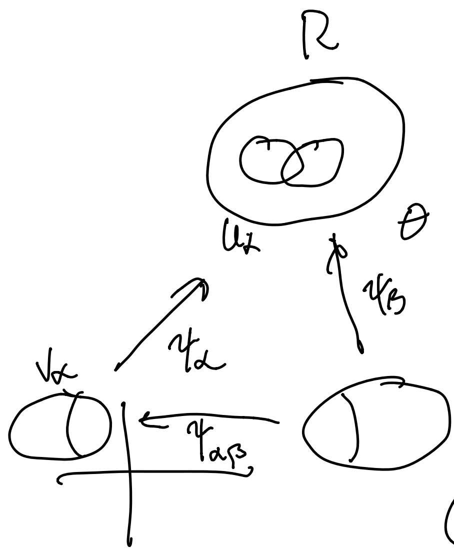
How do we define forms on surfaces?

Say $R$ is a surface (not nec. a Riemann surface) and $\mathcal{A}$ is an atlas for $R$.

If there was a 1-form $\theta$ on $R$ then for each inverse chart $\psi_\alpha$ there would be a 1-form $\theta_\alpha = \psi_\alpha^*(\theta)$ on $V_\alpha$. Since




$$\psi_\alpha \circ \psi_{\alpha\beta} = \psi_\beta$$

it would be the case that

$(\psi_\alpha \circ \psi_{\alpha\beta})^* = \psi_\beta^*(\theta)$ but

$(\psi_\alpha \circ \psi_{\alpha\beta})^* = \psi_{\alpha\beta}^*\left(\psi_\alpha^*(\theta)\right)$ so

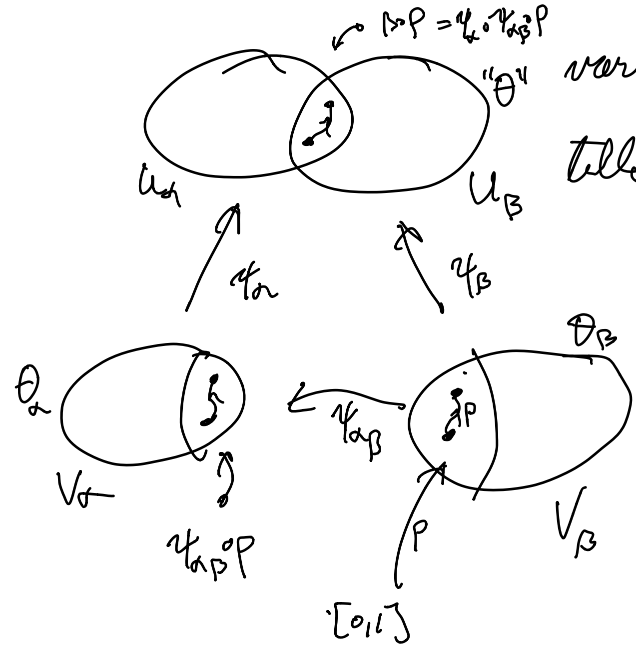
$$\psi_{\alpha\beta}^*(\psi_\alpha^*(\theta)) = \psi_\beta^*(\theta) \text{ or}$$

$$\psi_{\alpha\beta}^*(\theta_\alpha) = \theta_\beta$$

(where this makes sense).

If we wanted to do something with this 1-form like integrate over a short path then we could integrate in a coordinate chart.

The choice of the coordinate chart would not make a difference since the "change of θ variables formula" tells us that




$$\int_\rho \psi_\beta^*(\theta) = \int_{\psi_\beta(\rho)} \theta$$

$$= \int_{\psi_{\alpha\beta}(\rho)} \psi_\alpha^*(\theta)$$

or strictly in terms of $\theta_\beta, \theta_\alpha$

$$\int_\rho \theta_\beta = \int_{\psi_{\alpha\beta}(\rho)} \theta_\alpha$$

(since $\theta_\beta = \psi_{\alpha\beta}^*(\theta_\alpha)$).

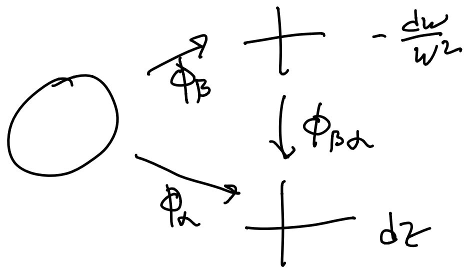
In other words we could deal with $\theta$ indirectly just using the forms $\theta_\alpha$... and the result will be independent of the particular choices we make.

We can now reverse this logic and say that a form on R *is* a collection of forms $\{\theta_\alpha\}$ which satisfy the consistency condition that $\theta_\beta = \psi_{\alpha\beta}^*(\theta_\alpha)$ where this makes sense (or $\theta_\beta = (\phi_{\beta\alpha}^{-1})^* \theta_\alpha$).

Now say that we have a Riemann surface R so that the change of variable maps $\psi_{\alpha\beta}$ are holomorphic. In this we say that a complex valued 1-form $\theta$ is meromorphic if each $\theta_\alpha$ can be written as $f_\alpha(z)\,dz$ where $f_\alpha$ is a meromorphic function on $V_\alpha$ $\alpha \in A$.

In this case $\psi_{\alpha\beta}^*(f_\alpha(z)\,dz) = f_\alpha(\psi_{\alpha\beta}(z)) \cdot \frac{d\psi_{\alpha\beta}}{dz} \cdot dz$ so the result of pulling back a meromorphic form $\theta$ is automatically a meromorphic form (since $\frac{d\psi_{\alpha\beta}}{dz}$ is holomorphic). In other words if a form is (locally) meromorphic in one chart it will be locally meromorphic in any other chart.

# Meromorphic forms on $\mathbb{C}_\infty$.

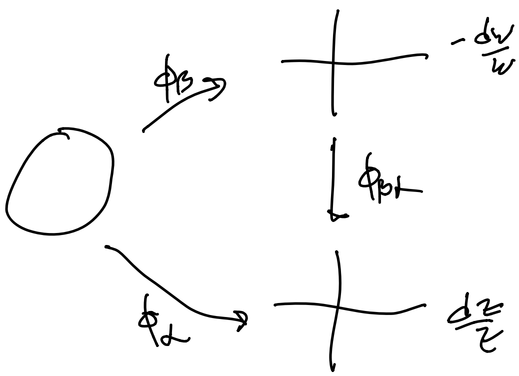



$$\phi^*_{\beta\alpha}(dz) = d\,\phi^*(z) = d\left(\tfrac{1}{w}\right) = -\frac{1}{w^2}\,dw.$$

$$z = \frac{1}{w}$$

$$dz = -\frac{1}{w^2}\,dw$$

Note 1 pole of order 2.



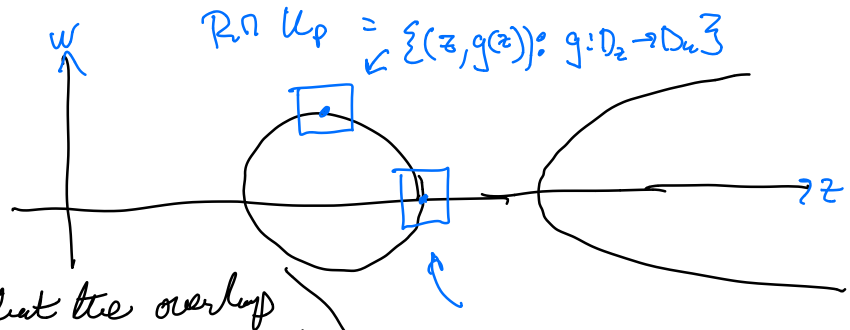

$$\phi^*_{\beta\alpha}\left(\frac{dz}{z}\right) = \frac{-\frac{1}{w^2}\,dw}{1/w} = -\frac{dw}{w}$$

Note pole of order 1 at 0 and order 1 at $\infty$.

We say that a meromorphic 1-form is <u>holomorphic</u> if it has no poles.
(Note that this property is independent of the system of charts used.)

Example. Riemann surfaces coming from varieties (defined by polynomial equations in $\mathbb{C}^2$).

In the discussion of hyper-elliptic $w^2 = P(z)$ surfaces I did not finish building the atlas. $\varphi_p(z) = (z, g(z))$.




(Check that the overlap functions are holomorphic.)

$\varphi_q(w) = (h(w), w)$

I carefully constructed the functions "$g$" but I did not construct the "$h$".

I gave an integral formula for $g$ which was helpful in understanding the global to topology of $R$. If we just want to prove the existence of $g$ we can write $g(z) = \sqrt{P(z)}$ where $\sqrt{\cdot}$ is a choice of a square root. Thus $\psi(z) = (z, g(z)) = (z, \sqrt{P(z)})$

$$= (z, w)$$

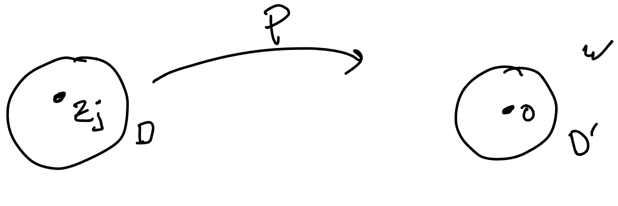
where $w^2 = P(z)$.

These charts cover all of $R$ except the points $(z_j, 0)$ where $P(z_j) = 0$.

Assumption: The roots of $P$ are simple.

When $P(z) = 0$ we have that $P'(z) \neq 0$ since the roots are simple. This implies that $P$ is locally invertible.

$w^2 = P(z)$



Let $\underset{w}{h: D'} \longrightarrow \underset{z}{D}$ be $P^{-1}$. Thus $P(h(w)) = w$ for $w \in D'$.

$$w^2 = P(z) \qquad z = h(w)$$

Claim $R \cap (D' \times D) = \{ \underset{\substack{\parallel \\ (z,w)}}{(h(w^2), w)} : w \in D' \}$

$$z = h(w^2)$$

Note that $P(z) = P(h(w^2)) = w^2$.

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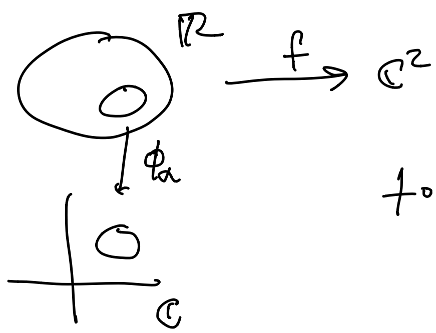
Def. We say that a map from $\varphi : U \overset{o}{\subset} \mathbb{C}$ to $\mathbb{C}^2$

$$\varphi(u) = (\varphi_1(u), \varphi_2(u))$$

is holomorphic if each coordinate function is holomorphic. We say a map from a Riemann surface $f: R$ to $\mathbb{C}^2$ is holomorphic if in each coordinate chart




$f \circ \phi_\alpha^{-1}$ is holomorphic.

This atlas gives a Riemann surface structure which is compatible with the complex structure on $\mathbb{C}^2$. In that the inclusion $R \to \mathbb{C}^2$ is holomorphic

Connection between elliptic integrals and elliptic surfaces.

Consider $\int \frac{dz}{\sqrt{1-z^2}}$ or $\int \frac{dz}{\sqrt{P(z)}}$ more generally

$z^2+w^2=1$

$w^2=1-z^2$

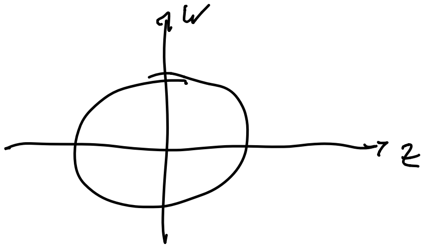
$w=\sqrt{1-z^2}$

When $\deg P = 2$ the anti-derivative can be computed in terms of inverse trig functions. When $\deg P = 3$ or $4$ these integrals arise in connection with computing the arclength of an ellipse hence the name "elliptic integral".

not completely well defined since $\sqrt{\ }$ has two branches. Becomes well defined after we choose a branch. Branch issues are exactly what Riemann surfaces deal with.

Let $w = \sqrt{1-z^2}$ then the integral becomes

$\int \frac{dz}{w}$ where $w$ is still not well defined.




Write $w^2 = 1-z^2$ then

We can restrict the function $w$ to

$R = \{(z,w) : w^2 = 1-z^2\}$.

We can consider the 1-form $\frac{dz}{w}$ restricted to $R$. If we choose a branch of $\sqrt{\ }$ then we get a coordinate chart $\varphi : z \mapsto (z, \sqrt{1-z^2})$

and $\varphi^*\left(\frac{dz}{w}\right)$ is

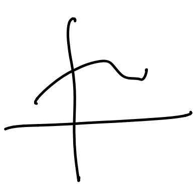
$\int \frac{dz}{\sqrt{1-z^2}}$ is

$\varphi : u \to z$

$\varphi^*(dz) = \frac{d\varphi}{du} \cdot du$ $\qquad dz = \frac{dz}{du} \cdot du$

Formula $\qquad u \mapsto (\varphi_z(u), \varphi_w(u))$

$$\varphi^*(f\,dz + g\,dw) = f(\varphi_z(u), \varphi_w(u)) \cdot \frac{d\varphi_z}{dz} \cdot dz + g(\varphi_z(u), \varphi_w(u)) \cdot \frac{d\varphi_w}{dw}\, dw.$$



Let $\pi_1, \pi_2$ be the complex coordinate functions on $\mathbb{C}^2$. $\pi_1(z,w)=z$, $\pi_2(z,w)=w$.

Let $dz = d\pi_1$ and $dw = d\pi_2$

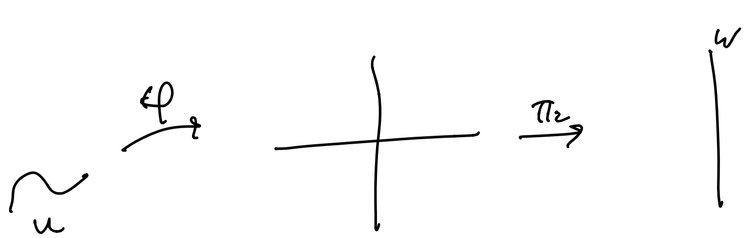
We define meromorphic 1-forms on $\mathbb{C}^2$ to be complex valued 1-forms that can be written as

$$f(z,w)\,dz + g(z,w)\,dw$$

where $f, g$ are meromorphic.

$$\varphi = (\varphi_1, \varphi_2) \qquad \varphi(u) = (\varphi_1(u), \varphi_2(u))$$

$$\varphi^*(f\,dz + g\,dw) =$$



$$\downarrow \pi_1$$

$$\underline{\qquad\qquad} \; z$$

uses $z = \pi_1$

$$\varphi^*(dz) = d(\varphi^* z) = d(\pi_1 \circ \varphi) = d(\varphi_1)$$

$$\varphi^*(dz) = d(\pi_1 \circ \varphi) = d(\varphi_1) = \frac{d\varphi_1}{du}\, du$$

$$= d(\pi_1 \circ \varphi(u)) = d(\varphi_1(u)) = \frac{d\varphi_1}{du} \cdot du.$$

$$\varphi^*(dw) = d(\pi_2 \circ \varphi) = d(\varphi_2) = \frac{d\varphi_2}{du} \cdot du.$$

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$$\varphi^*(dz) = d(\varphi^*(z)) = d(\pi_1 \circ \varphi)$$

pulling back a function is composition.

$$= d(\varphi_1)$$

$$= \frac{d\varphi_1}{du}\, du.$$

$$\varphi^*(dw) = \frac{d\varphi_2}{du}\, du.$$

Check: The pullback of a meromorphic form on $\mathbb{C}^2$ to $U^{c\varphi}$ under a holomorphic map is meromorphic.

Using the language of Riemann surfaces allows us to translate the ambiguity of the integrand $\int \frac{dz}{\sqrt{1-z^2}}$ into the ambiguity of the choice of a chart for a <u>well defined</u> 1-form on the Riemann surface $R$.

Is this an advantage?

The points $z = \pm 1$ look bad from the viewpoint of $\frac{1}{\sqrt{1-z^2}}$. Looks like a pole.

What does it look like on the surface?

$dz$ and $dw$ are linearly independent holomorphic 1-forms on $\mathbb{C}^2$. When we restrict them to $R$ they are no longer linearly independent. What relation do they satisfy?

Claim on $R$ we have $w^2 = P(z)$

Consider the function $f(z,w) = P(z) - w^2$ on $\mathbb{C}^2$.

Restricted to $R$ this function is zero

so $df = 0$ but $df = d\,P(z) - dw^2$

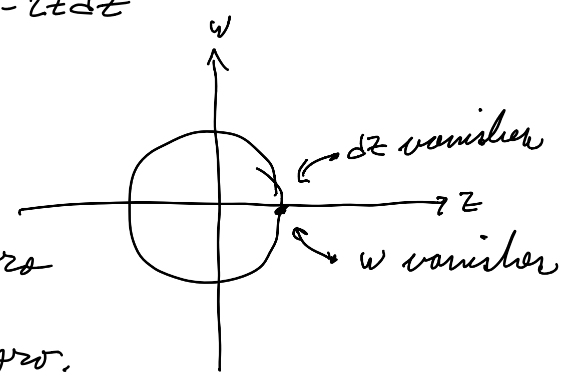
$$= P'(z)dz - 2w\,dw.$$

On $R$ we have $2w\,dw = P'(z)\,dz$.

$P(z) = 1 - z^2$ $\qquad 2w\,dw = -2z\,dz$

$w\,dw = z\,dz$

$$\frac{dw}{z} = \frac{dz}{w}$$




$$\frac{dz}{w} = \frac{dw}{z}$$

non-zero

non-zero.

perfectly good non-zero holomorphic 1-form.

In general $\quad 2w\,dw = P'(z)\,dz$

non-zero.

$$2\frac{dw}{P'(z)} = \frac{dz}{w}$$

non-vanishing.

Our original problem becomes a problem of integrating a non-vanishing 1-form on R.

Definition of the integral on a Riemann surface.

Definition of the exterior derivative on a Riemann surface. Independent of choice of chart.

F is an anti-derivative of $\theta$ if $DF = \theta$.

Anti-derivatives are well defined up to the addition of a constant.

Definition of a zero or pole on a Riemann surface.

$f\,dz \to f$ is well defined up to mult. by a non-zero holomorphic function.

The values of $f$ are not well defined but the orders of zeros or poles are well defined.

If $\theta$ is a non-zero hol. 1-form then $F$ so that $dF=\theta$ gives a local chart for $R$. Collection of such local charts gives an atlas. Overlap functions for this atlas have the form $z \mapsto z+c$ (where $c$ corresponds to the ambiguity in the constant of integration).

(Regular atlas not inverse atlas.)

Define the period map from $\pi_1(R) \to \mathbb{C}$ by integrating along loops.

On $\tilde{R}$ we can put these charts together to get a single function well defined up to a constant.

dev : $\tilde{R} \longrightarrow \mathbb{C}$.

Action of the deck group on $\tilde{R}$ corresponds to the period homomorphism.

If the developing map is bijective then the inverse map is periodic.

Example.

$$R = \{ (z,w) : zw = 1 \}.$$

$$z\,dw = w\,dz$$

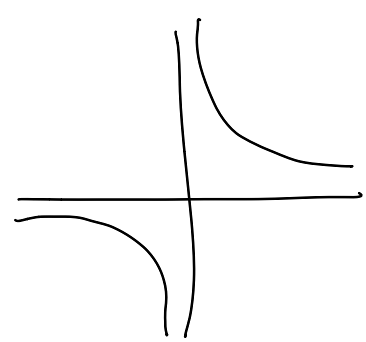
$$\frac{dw}{w} = \frac{dz}{z}.$$

$\theta = \frac{dw}{w} = \frac{dz}{z}$ is non-zero

$dz$ is non-zero on $R$.

$dz$ gives a charts

$z$ is the anti-derivative



$z$ is well defined on $R$, no need to lift.
Image of $R$ under $z$ is $\mathbb{C}-\{0\}$.
Developing map is not invertible.

In general a meromorphic $\theta$ on $\mathbb{C}^2$ need not be closed. It will be closed when restricted to $R$.

Where it is non-zero a Holomorphic 1-form on $\mathbb{C}^2$ determines a 1-complex dimensional subspace of $\mathbb{C}^2$.

Hol. 1-form gives a function from tangent vectors to $\mathbb{C}$ which is complex linear.

Can use this observation to determine

the tangent space to a variety.

At a both $p$ $dz$, $dw$ pullback to forms that vanish at $p$.

$$\frac{dz}{\sqrt{1-z^2}}$$ represents the pullback of $\frac{dz}{w}$ to the $z$ plane using one of the coordinate charts $z \mapsto (z, \sqrt{1-z^2})$.