Recall that we introduced complex valued forms last time and I claimed that this was a natural language for complex analysis. I will prove 2 results which make this connection.

Example. Any $f = u + iv$ is a <u>holomorphic</u> function then $df = \frac{\partial f}{\partial z} dz$ (as $\mathbb{C}$ valued 1-forms).

$$df = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + i \frac{\partial v}{\partial x} dx + i \frac{dv}{\partial y} dy$$

$$= \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) dx + \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) dy$$

$$= \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) dx + i \left(\frac{\partial v}{\partial y} - i \underset{\updownarrow \text{CR}}{\frac{\partial u}{\partial y}} \right) dy$$

$$= \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) dx + i \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial y} \right) dy$$

$$= f' dx + i \cdot f' dy$$

$$= f'(dx + i dy) = f' dz.$$

$$= \frac{\partial f}{\partial z} \cdot dz.$$

CR:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}.$$

Note that the derivative (from $\mathbb{C}$-analysis) now appears in the form of a 1-form. The 1-form makes explicit reference to a particular coordinate z. (Relevant when we want to change our local coordinate.) Like Leibniz notation the result looks obvious but hides some actual content.

Note that the natural context for asking whether a particular function g has an anti-derivative is asking whether $g\,dz$ is exact.

If $g\,dz = df$ then $f' = g$.

Theorem. $d(f\,dz)=0$ iff f satisfies the Cauchy-Riemann equations.

$$d(f\,dz) = df \wedge dz = \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) \wedge (dx + i\,dy)$$

$$\begin{aligned}
&= \left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right)\wedge dx \\
&\quad + i\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)\wedge dx \\
&\quad + i\left(\frac{\partial u}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)\wedge dy \\
&\quad - \left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)\wedge dy
\end{aligned}$$

$$\begin{aligned}
&= \frac{\partial u}{\partial y}\,dy\wedge dx \\
&\quad + i\,\frac{\partial v}{\partial y}\,dy\wedge dx \\
&\quad + i\,\frac{\partial u}{\partial x}\,dx\wedge dy \\
&\quad - \frac{\partial v}{\partial x}\,dx\wedge dy
\end{aligned}$$

$$= \left(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)dx\wedge dy + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\wedge dy$$

= 0 since:

the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

So if g is holomorphic then $g\,dz$ is closed. g' has an anti-derivative if the <u>closed</u> form $g\,dz$ is <u>exact</u>.

Math 3510 Day 28 pullback commutes with d.

Naturality of the exterior derivative.
(also important for manifolds.)

Theorem:

Day 29 0:00

$$d(G^*\omega) = G^*(d\omega)$$

Function case

$$d(G^*f) = G^*(df)$$

$$\parallel$$

$$D(f \circ G)(\vec{a}) = Df$$

Function case:

$$d(G^*f)(V) = D_V G^*f$$

$$= D_V f \circ G$$

$$= Df \circ DG(V)$$

$$G^*(df)(V)$$

$$G = \underbrace{\begin{bmatrix} G_1 \\ \vdots \\ G_m \end{bmatrix}}_{n} \Big\} m$$

Remark:

$G^*(u_j) = G_j$

$G^*(du_j) = dG_j$.

"$G_j = u_j$" means

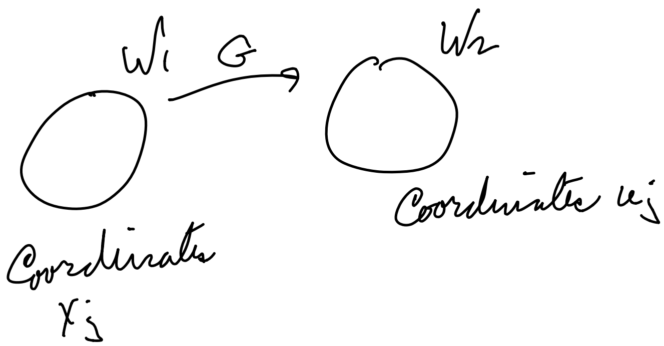
"$G_j = G^* u_j$"

G_j $G^*(u_j)$

Want to show that

"$dG_j = du_j$" means $dG_j = G^*(du_j)$.

$$G^*(d\theta) = d(G^*(\theta)) \text{ for arbitrary } \theta.$$



Now consider θ. Want to show

$$G^*(d\theta) = d(G^*\theta).$$

If we write $\theta = \sum_I f_I dx_I$. Suffices to prove the result for each component function since both sides are linear.

So consider $\theta = f dx_I$.

$$G^*(d\theta) = G^*(df \wedge dx_I) = G^*(df) \wedge G^*(du_I)$$

$$= d(G^*f) \wedge G^*(du_{i_1} \wedge \dots du_{i_k})$$

$$= d(G^*f) \wedge G^*(dx_{i_1}) \wedge \dots G^*(dx_{i_k})$$

$$= d(G^*f) \wedge d(G^*x_{i_1}) \wedge \dots d(G^*x_{i_k})$$

$$= d(G^*f) \wedge du_{i_1} \wedge \dots du_{i_k}$$

$$\stackrel{?}{=} d(G^*f \wedge du_{i_1} \dots du_{i_k})$$

$$= d(G^*(f dx_I))$$

$$d(G^*f \wedge du_{i_1} \dots du_{i_k})$$

$$= d(G^*f) \wedge (du_{i_1} \wedge \dots du_{i_k})$$

$$= d(G^*f) \wedge \Big(d^2u_{i_1} \wedge du_{i_2} \wedge \dots du_{i_k}$$
$$- u_{i_1} \wedge d^2u_{i_2} \wedge \dots du_{i_k}$$
$$\pm$$
$$\pm$$
$$\pm u_{i_1} \wedge u_{i_2} \wedge \dots d^2u_{i_k} \Big)$$

$$= d(G^*f) \wedge (du_{i_1} \wedge \dots du_{i_k})$$

$(F \circ G)(f)$

$(F \circ G)^*(\theta) = F^*(G^*(\theta))$. Works for functions!

Works for 1 forms since 1-forms are df

One highlight of a course on forms is a major generalization of the fundamental theorem of calculus called the generalized Stokes Theorem. This works in all dimensions and for manifolds and has the form:

$$\int_M d\omega = \int_{\partial M} \omega .$$

manifold with boundary $= \partial M$.

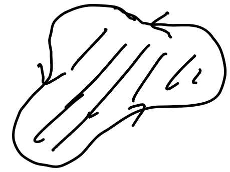
At the moment we only need a 2-dim version which I am assuming is familiar:

Gauss' Theorem. Let $U \subset \mathbb{R}^2$ be a subsurface with boundary and ω a 1-form then:

$$\int_U d\omega = \int_{\partial U} \omega$$

inherits an orientation

Choose a direction on ∂U so that when facing forward your left hand points into U.



If $\omega = a\,dx + b\,dy$ then $\partial\omega = \frac{\partial a}{\partial y} dy \wedge dx + \frac{\partial b}{\partial x} dx \wedge dy$

$$= \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx \wedge dy$$

So $\boxed{\int_U \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx \wedge dy = \int_{\partial U} a\,dx + b\,dy.}$

Example:

Second criterion for exactness:

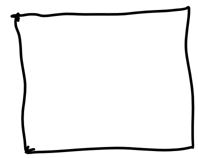
Homotopy invariance. Winding number

Thm.

any homeo. to a disk

If U is simply connected then and ω is a 1-form with $d\omega = 0$ on U then $\omega = df$.

Application



Let $\gamma_0, \gamma_1 : [a,b] \longrightarrow X$ be two curves with the same starting and ending points z_0 and z_1. Then γ_0 and γ_1 are <u>homotopic</u> if there exists a map h of the rectangle $[a,b] \times [0,1]$ into X with

$$h(t,0) = \gamma_0(t)$$
$$h(t,1) = \gamma_1(t)$$
$$h(a,s) = z_0$$
$$h(b,s) = z_1$$

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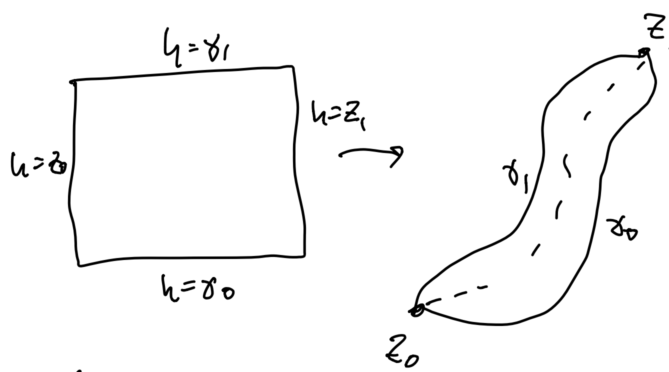


Claim. If $U \subset \mathbb{R}^2$ and ω is a 1-form with $d\omega = 0$, if γ_0 and γ_1 are homotopic with h smooth then

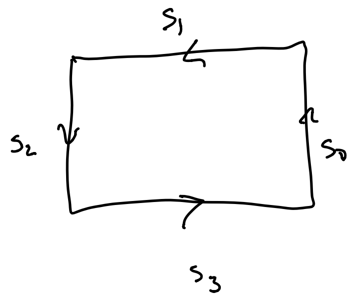
$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega .$$

Proof. Consider $h^*\omega$ on B.

We have $\displaystyle\int_B d(h^*\omega) = \int_B h^* d\omega = \int_B 0 = 0.$

So by Green's thm.

$$\int_{\partial B} h^*\omega = \int_B d(h^*\omega) = 0$$



$$\cancelto{0}{\int_{s_0} h^*\omega} + \int_{s_1} h^*\omega + \cancelto{0}{\int_{s_2} h^*\omega} + \int_{s_3} h^*\omega = 0$$

$$\int_{s_1} h^*\omega + \int_{s_3} h^*\omega = 0$$

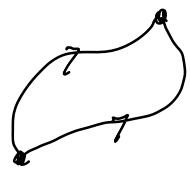
$$\downarrow \qquad\qquad \downarrow$$

$$-\int_{\gamma_0} \omega + \int_{\gamma_1} \omega = 0$$

So $\displaystyle\int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$

Cor. If ω is a closed $\mathbb{R}$-or-$\mathbb{C}$ valued form on U then ω defines a homomorphism from $\pi_1(U) \to \mathbb{R}$ or $\mathbb{C}$.

Cor. If ω is closed 1-form and the induced map is trivial then ω is exact.

Proof.  In this case $\oint_\gamma \omega = 0$ for any loop ω.

(Since any loop is homotopic to a constant loop.) This is one of our criteria for exactness.