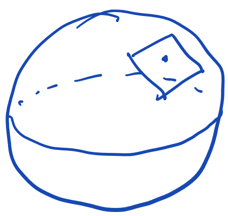
A Riemann surface is a fairly abstract object (like a number of things in modern mathematics).

How did Riemann and Weyl come up with this definition?

How do we make sense of it?
In what way is there geometry attached to a Riemann surface?

As in last class let S^2 be the unit sphere in $\mathbb{R}^3$. Let $p \in S^2$. Let T_p be the tangent plane to S^2 at the point p.



We can identify T_p with

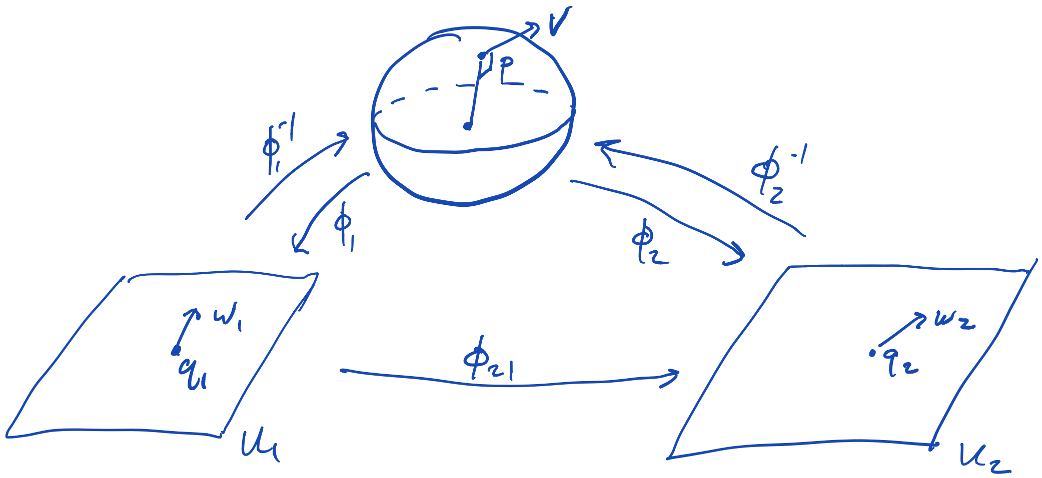
$$T_p = \{ V \in \mathbb{R}^3 : p \cdot V = 0 \}.$$

The tangent bundle of S^2 is the collection of tangent planes:

$$T(S^2) = \{ (p, V) : p \in S^2, V \in \mathbb{R}^3 \ p \cdot V = 0 \}.$$

Note that T_p is a vector space so that $T(S^2)$ is a disjoint union of vector spaces.

How does the vector space structure interact with the Riemann surface structure on S^2?



$\phi_1^{-1} : \mathbb{C} \longrightarrow \mathbb{R}^3$ $\qquad$ $\phi_2^{-1} : \mathbb{C} \longrightarrow \mathbb{R}^3$

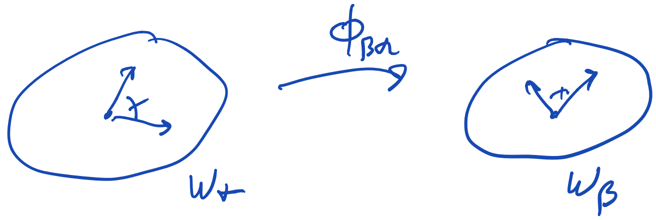
Say $\phi_1(p) = q_1$. $D\phi_1^{-1}$ takes a tangent vector w at q_1 to a tangent vector at p. Say $\phi_2(p) = q_2$. $D\phi_2^{-1}$ takes a tangent vector at q_2 to a tangent vector at p. By the chain rule $D\phi_{21}$ takes w_1 to w_2.

Let $T(U_1) = \{(q_1, w_1) : q_1 \in U_1, w_1 \in \mathbb{C}\}$
$T(U_2) = \{(q_2, w_2) : q_2 \in U_2, w_2 \in \mathbb{C}\}$.

Note that we can reconstruct the tangent bundle $T(S^2)$ from $T(U_1)$ and $T(U_2)$ by using identifications induced by ϕ_{12}.

For <u>any</u> Riemann surface R we can construct an abstract tangent bundle $T(R)$ by gluing together tangent bundles of charts as in this example. In the special case of S^2 we can identify this abstract tangent bundle with a concrete tangent bundle of a smooth surface in $\mathbb{R}^3$.

Remark. Given two tangent vectors v, w in $T_p \subset T(R)$ it makes sense to talk about the angle between them.



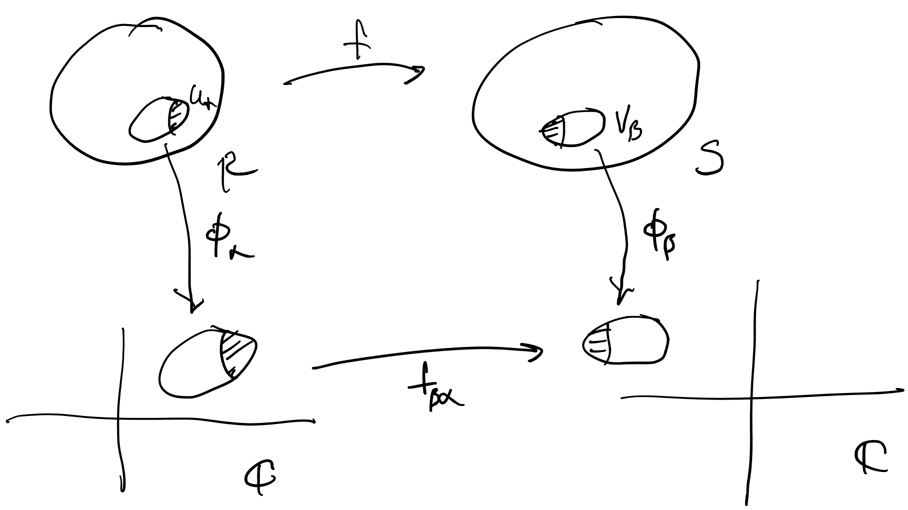
Since $\phi_{\beta\alpha}$ is holomorphic it preserves angles between vectors so the angle is independent of the coordinate chart.

This is an example of how we think about and talk about Riemann surfaces. A property makes sense for R if it makes sense in any coordinate chart.

Definition. (Holomorphic function between Riemann surfaces)

Let R and S be Riemann surfaces with atlases $\{(\phi_\alpha, U_\alpha)\}$ and $\{(\psi_\beta, V_\beta)\}$.

Let $f: R \to S$ be a function.



For $U_\alpha \subset R$ and $V_\beta \subset S$ we set

$f_{\beta\alpha} = \psi_\beta \circ f \circ \phi_\alpha^{-1}$ (The domain is the set on which this expression makes sense. It can be empty.)

$f_{\beta\alpha} : U \overset{\subset \mathbb{C}}{\longrightarrow} \mathbb{C}$

We say that f is <u>holomorphic</u> if each function $f_{\beta\alpha}$ is holomorphic.

Example: We have defined a map $f = \phi_1^{-1}$ from $\mathbb{C} \xrightarrow{f} S^2$ where both of these are Riemann surfaces. ($\mathbb{C}$ is an open subset of $\mathbb{C}$). Claim that f is holomorphic. In order to check this we need to

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "f"] \arrow[d, "\psi_0 = \mathrm{id}"] & S^2 \arrow[d, "\phi_1"] \\
\mathbb{C} \arrow[r] & \mathbb{C}
\end{tikzcd}$$

check that $\phi_1 \circ f \circ \psi_0^{-1}$ is holomorphic and that $\phi_2 \circ f \circ \psi_0^{-1}$ is holomorphic.

$\phi_1 \circ \phi_1^{-1} \circ \mathrm{id}\,(z) = z$ ok.

$\phi_2 \circ \phi_1^{-1} \circ \mathrm{id}\,(z) = \frac{1}{z}$ ok.

Remark. Consider $U^{\subset \mathbb{C}} \xrightarrow{f} V^{\subset \mathbb{C}}$. A priori we have defined two different notions of what it means to be a holomorphic function from U to V. We have the classic definition and the Riemann surface definition. It is easy to check that these two definitions are the same.

Example 3. Let λ and μ be complex numbers which are linearly independent over $\mathbb{R}$.

Let $\Gamma = \{m\lambda + n\mu : m, n \in \mathbb{Z}\}$ (We call Γ a lattice.) Γ is a group and it acts on $\mathbb{C}$ by addition $\gamma = (m\lambda + n\mu)$ and $z \in \mathbb{C}$

$\gamma(z) = m\lambda + n\mu + z$.

Let $\mathbb{C}/\Gamma$ denote the quotient space of this action and let $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ be the quotient map. $z \sim z'$ if $z = z' + \gamma$ for Γ. A set $U \subset \mathbb{C}/\Gamma$ is open iff $\pi^{-1}(U)$ is open in $\mathbb{C}$.

Warning. In general the quotient space of a topological space by the action of an infinite group can be very nasty.
Let us show that in this case it is nice.

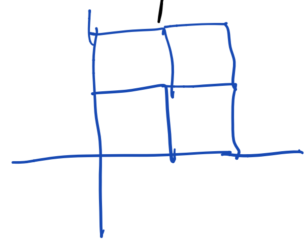
Step 1. If $\Gamma = \{ m + ni : m, n \in \mathbb{Z} \}$ then we can identify $\mathbb{C}/\Gamma$ with $\mathbb{R}^2/\mathbb{Z}^2 = (\mathbb{R}/\mathbb{Z}) + (\mathbb{R}/\mathbb{Z})$ $= S^1 \times S^1$.

Step 2. For general $\Gamma = \{ m\lambda + n\mu \}$ write λ and μ as column vectors in $\mathbb{R}^2$ and let $\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$ $\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ and let $A = \begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix}$.

$A(\mathbb{Z}^2) = \Gamma$. Since λ and μ are linearly independent over $\mathbb{R}$ the matrix A is invertible and A induces a homeomorphism:

$$\begin{array}{ccc} (\mathbb{C}=)\mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2\ (=\mathbb{C}) \\ \downarrow & & \downarrow \\ \mathbb{R}^2/\mathbb{Z}^2 & \longrightarrow & \mathbb{R}^2/\Gamma \end{array}$$

$\mathbb{R}^2 \xrightarrow{\pi} T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is a covering map.



The homeomorphism induces a homeomorphism from $\mathbb{R}^2/\mathbb{Z}^2$ to $\mathbb{R}^2/\Gamma$. $=\mathbb{C}/\Gamma$

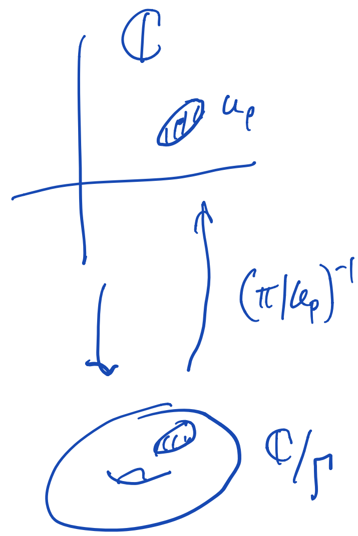
So we see that $\mathbb{C}/\Gamma$ is homeomorphic to a torus.

We construct an atlas on $\mathbb{C}/\Gamma$ as follows. Given $z \in \mathbb{C}$ we can find an open set U with $z \in U$ so that U is disjoint from all its images.

In $\mathbb{R}^2$ a disk of radius less than 1 has this property since the distance between two centers is greater than or equal to 1.

Let U_p be the image of such a disk under A.

Let $\phi_p = (\pi|U_p)^{-1}$.



Definition. We say that two Riemann surfaces R, S are holomorphically equivalent if there there is a holomorphic map $f: R \to S$ with a holomorphic inverse.

(Of course f is conformal so we sometimes say that R and S are conformally equivalent.)

We have constructed a family of Riemann surfaces all of which are topologically equivalent to tori and hence all topologically equivalent to each other.

We can ask whether they are conformally equivalent? The answer is no.

We have constructed a large number of conformally distinct surfaces.

$$\mathbb{C}^2 = \{(z,w) : z,w \in \mathbb{C}\}$$

$$W = \mathbb{C}^2 - \{(0,0)\}.$$

Define an equivalence relation $\sim$ on W by

$(z,w) \sim (u,v)$ iff for some $t \in \mathbb{C}$ $(z,w) = (tu, tv)$.

note $t \in \mathbb{C} - \{0\}$.

Remark that these equivalence classes are orbits of a group action where $\mathbb{C} - \{0\}$ is given the group structure coming from multiplication.

Write the equivalence class of (z,w) as $[z:w]$ (by definition z, w not both 0)

Let $\mathbb{CP}^1$ be the set of equivalence classes

$$\begin{tikzcd}
W \arrow[rr, "q"] \arrow[dr, "\alpha"'] & & \mathbb{CP}^1 \arrow[dl, "\beta"] \\
& \mathbb{C}_\infty &
\end{tikzcd}$$

$q((z,w)) = [z:w]$

$$\alpha((z,w)) = \frac{z}{w} \text{ if } w \neq 0$$
$$= \infty \quad w = 0.$$

$$\beta([z:w]) = \frac{z}{w} \quad w \neq 0$$
$$\infty \quad w = 0.$$

$\mathbb{C} \to S^2$. If we pull back the atlas on S^2 to $\mathbb{CP}^1$ we get $U_1 = \{[z:w] : w \neq 0\}$

$U_2 = \{[z:w] : z \neq 0\}$

$\varphi_1 : U_1 \to \mathbb{C}$ $\qquad \varphi_1([z:w]) = \frac{z}{w}$

$\varphi_2 : U_2 \to \mathbb{C}$ $\qquad \varphi_2([z:w]) = \frac{w}{z}$.

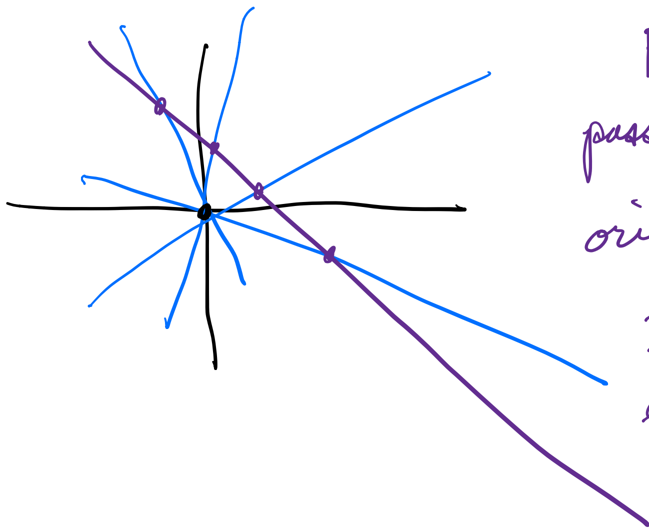
This gives $\mathbb{CP}^1$ a Riemann surface structure.

Could add additional charts $\varphi_*([z:w]) = \frac{az+bw}{cz+dw} \in \mathbb{C}$, defined where $cz+dw \neq 0$, $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$.

This construction of $\mathbb{CP}^1$ $(= S^2)$ suggests symmetries of $\mathbb{CP}^1$.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is any 2×2 complex matrix with $\det \neq 0$, then the map $\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix}$ preserves the previous atlas. In terms of the chart φ_1 above we have $\varphi_1^{-1}(z) = \begin{pmatrix} z \\ 1 \end{pmatrix} \xmapsto{A} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix}$

$\varphi_1 \circ A \circ \varphi_1^{-1}(z) = \frac{az+b}{cz+d}$ $\rightsquigarrow$ gives us a linear fractional transformation.



Pick a line $\ell+Q$ not passing through the origin. Note that this line intersects every line passing through 0 except ℓ.

We define a chart by mapping each ℓ' to $\ell \cap \ell'$.

$\varphi: \mathbb{C} \to \ell$

Related to projective geometry.

$\ell + p$, $\ell' + q$