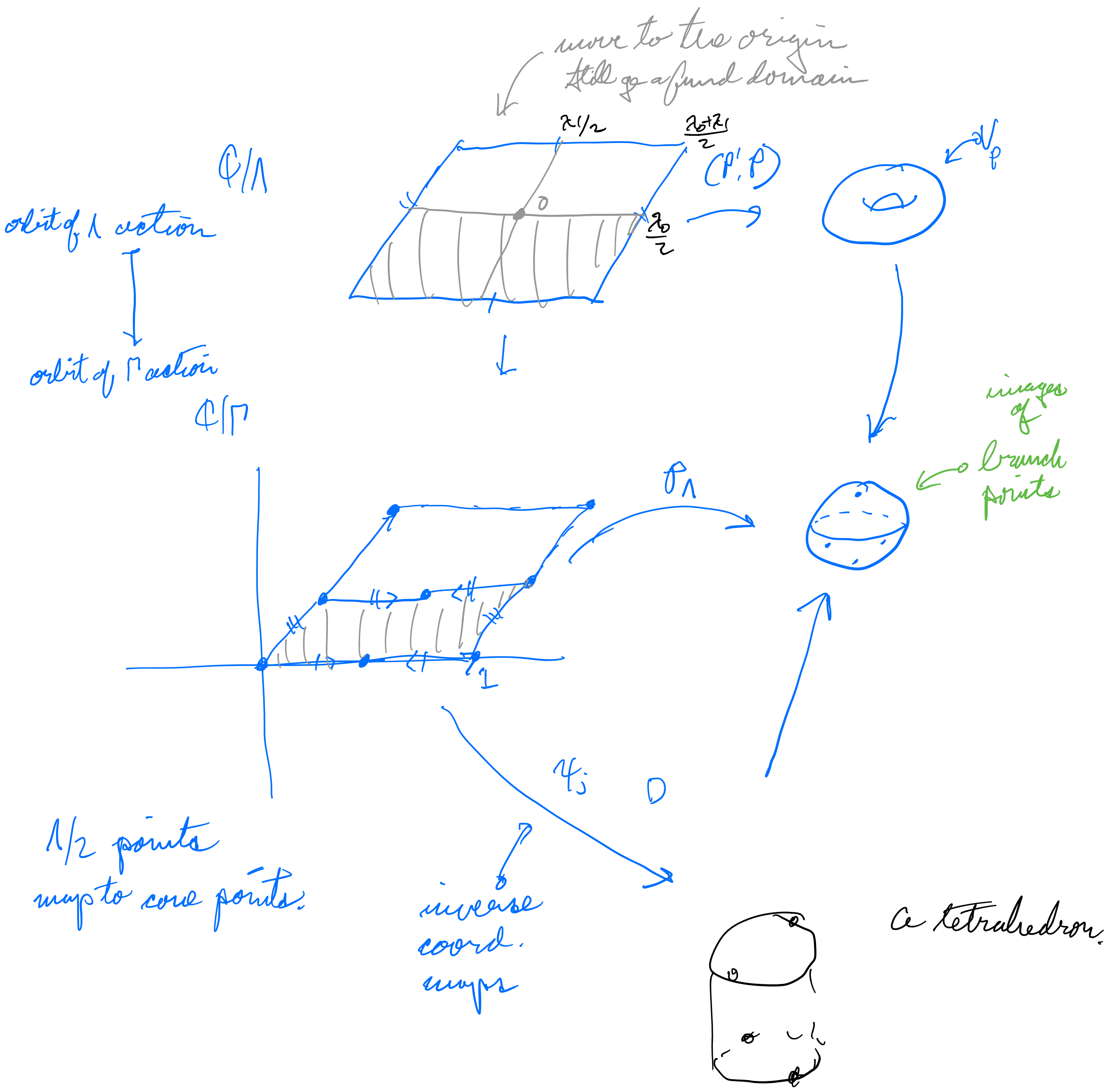
geometric object

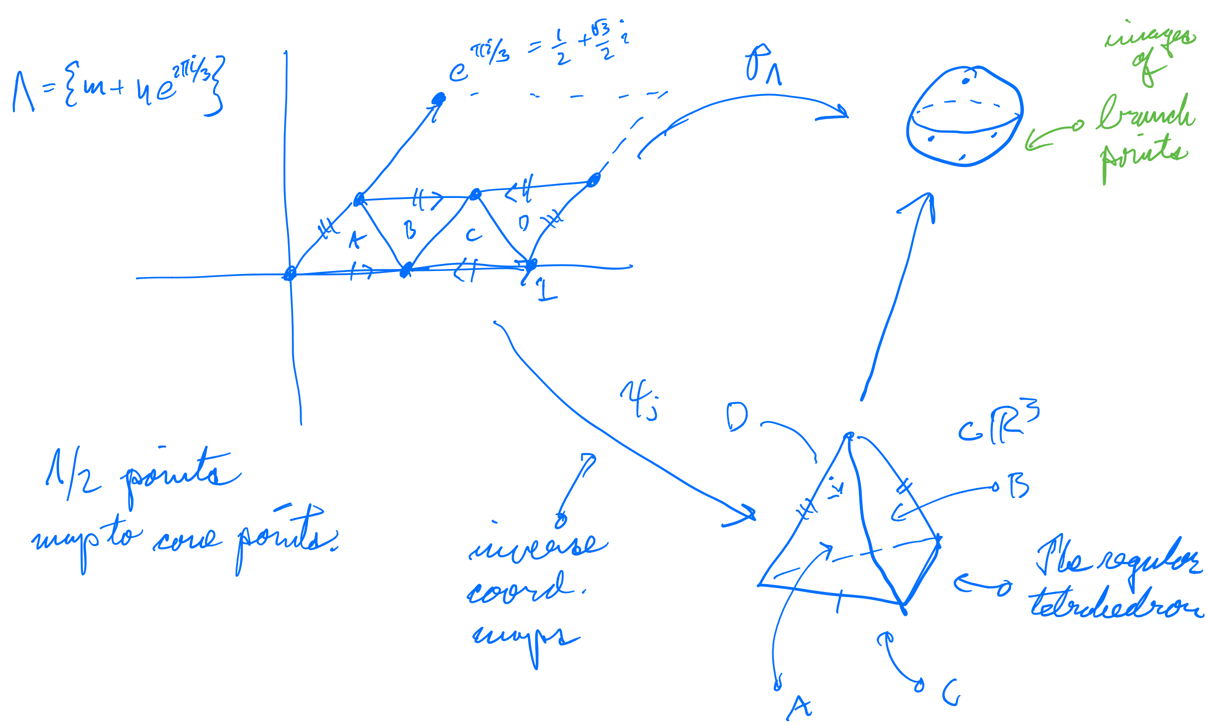
$$\begin{tikzcd}
\mathbb{C}/\Lambda \arrow[r, "\deg 2"] \arrow[d, dashed, "q", "\deg 2"'] \arrow[dr, "p"] & V_Q \arrow[d, "\deg 2"] \\
\mathbb{C}/\Gamma \arrow[r, "f"', "\deg 1"] & \mathbb{C}_\infty
\end{tikzcd}$$

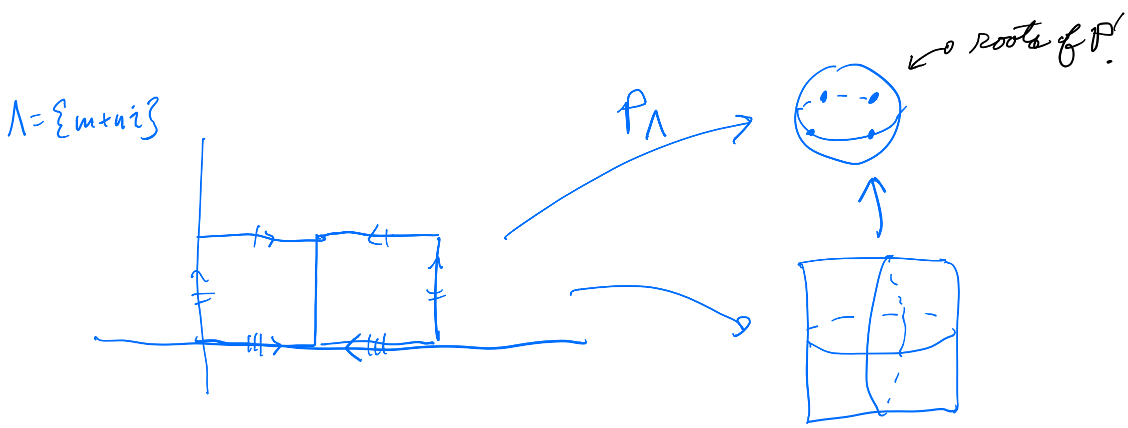
Claim that $\mathbb{C}/\Gamma$ can be given a Riemann surface structure obtained by gluing together triangles by isometries along edges. As in the case of boundaries of polyhedra this geometric structure will have cone points.

With this set up we can interpret the function $f: \mathbb{C}/\Gamma \to \mathbb{C}_\infty$ as giving a holomorphic equivalence between two different conformal structures built in two very different ways.

$$\Lambda \longrightarrow \Gamma \xrightarrow{\text{derivative}} \{\pm 1\}$$







Samosa versus ravioli

The pillowcase

Half-translation structure

Overlaps of the form $z \mapsto \pm z + c$.

More classical approach to half-translation structures: quadratic differentials.

Quadratic differentials are expressions of the form

$$f(z)\,dz^2 \qquad f(z)\,(dz)^2$$

interpret as / not as $d(z^2)$.

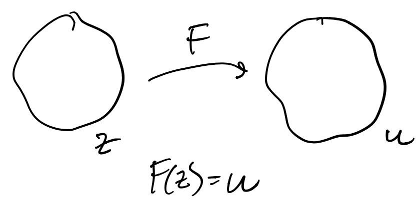
f is holomorphic

Can think of these as functions on tangent vectors

$$f(z)(dz)^2\,(\dot{v}_z) = f(z)\,\dot{v}_z^2.$$

Quadratic differentials give <u>quadratic functions</u> (complex valued) on tangent vectors

We can also think of quadratic differentials in terms of their transformation properties:



$$F^*(du^2) = \left(\frac{dF}{dz}\right)^2 dz^2.$$

(This point of view allows you to define quadratic differentials on Riemann surfaces)

A hol 1-form $f(z)\,dz$ gives rise to a quadratic differential by squaring it:

$f(z)\,dz \rightsquigarrow f^2(z)\,dz^2$.

Language of quadratic differentials allows us to turn the expression

$\displaystyle\int \frac{dz}{\sqrt{Q(z)}}$ into a well defined object on $\mathbb{C}$

namely look at the square of the 1-form $\frac{dz}{\sqrt{Q(z)}}$.

We get $\frac{dz^2}{Q(z)}$.

There is a 1-1 correspondence between quadratic differentials and half-translation structures.

Zeros/poles of order k for the quadratic differential correspond to cone points with cone angle $(k+2)\pi$.

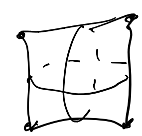
k	cone angle
-1	π
0	2π
1	3π

Maps of the form γ $z \mapsto \pm z + C$ preserve the quadratic differential dz^2 since the $\frac{d\gamma}{dz} = \pm 1$ $\left(\frac{d\gamma}{dz}\right)^2 = 1$.

Quadratic differential lines

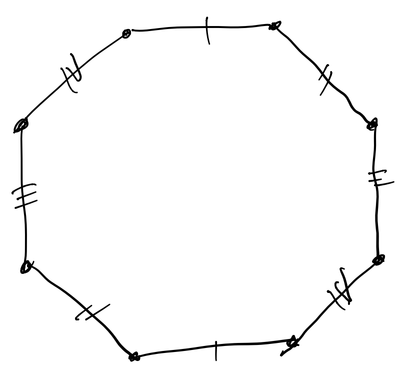
$$\int \frac{dz}{(z+1)(z-1)(z+2)(z-2)} \longrightarrow \text{4 simple poles}$$

Order $= k = -1$. Cone angle is π.



What happens in higher genus?

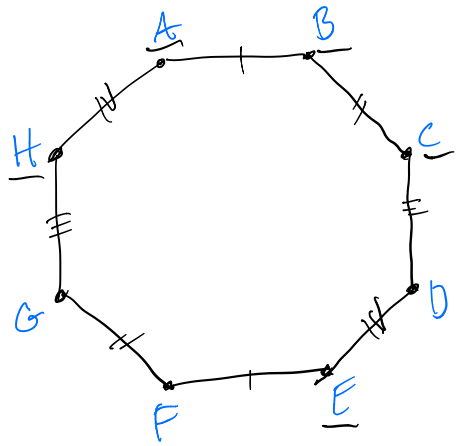
Genus 2 example:



As we have seen we get a Riemann surface structure on the resulting quotient space.

If we glue these sides together we form a Riemann surface with a well defined 1-form since the gluings respect the 1-form dz on $\mathbb{C}$.

When we glue the sides we induce certain identifications of vertices.



Side I gluing pairs A with F.

Side II gluing pairs F with C.

Side III pairs C with H

Side IV pairs H with E

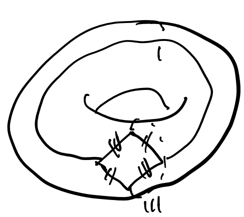
Side I pairs E with B.

Side II pairs B with G.

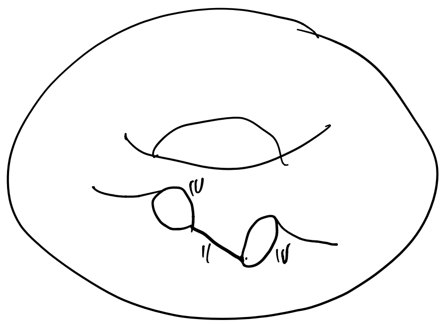
Result is that all vertices are identified to a single point.

The cone angle at that point is $\frac{3\pi}{4}\cdot 8 = 6\pi$.

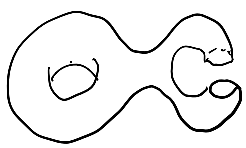
Topological picture: Identify pairs I and III



Now identify pair 2. Get a torus with 2 disks removed.



If we glue together these boundaries we get a surface of genus 2.



Prop. This Riemann surface corresponds to the algebraic curve with equation $w^2 = z(z^4-1)$.

The holomorphic 1-form that we have constructed corresponds to $\frac{dz}{w}$ on V_Q where $Q(z) = z(z^4-1)$.

The 1-form $\frac{dz}{w}$ is constructed from a quadratic differential $\frac{dz^2}{z(z^4-1)}$ on $\mathbb{C}_\infty$.

The situation corresponds to the hyper elliptic integral
$$\int \frac{dz}{\sqrt{z(z^4-1)}}.$$

The vertex of the octagon corresponds to the point at ∞ in V_Q.

(Secretly I know that every surface of genus 2 is hyper-elliptic.)

How do you prove this?

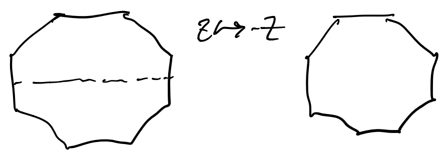
A characteristic feature of hyper-elliptic integrals is the fact that they are built from 1-forms which come from taking square roots of quadratic differentials.

This means that there is an involution of V_Q which takes the hol. 1-form to its negative.

In terms of the 1-form θ on the octagon there is an involution that takes θ to $-\theta$.

This is called the hyper-elliptic involution.

Start by finding the hyper-elliptic involution geometrically.



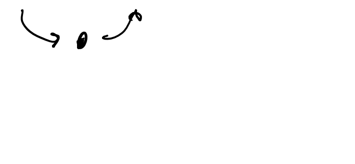
Find the fixed points of this involution.

There are 6 of these.

One at 0, one at each edge and one at the one point.

These fixed points will give rise to roots of Q. Note that there is an extra conformal symmetry which "rotates" the edge points. This commutes with the hyper-elliptic involution hence induces a conformal automorphism of $\mathbb{C}_\infty$ which leaves the set of roots invariant.

Only possibility is $z \mapsto iz$.



Q has these points as simple roots.
$\frac{dz^2}{Q}$ has these points as simple poles.

$\frac{dz^2}{Q}$ has a pole of 4 at ∞ corresponding to the cone point with cone angle 6π.

Remark. For the elliptic curve of genus 1 we have a map given by integration from $V_Q \to$ torus. We don't have this in genus 2 but if choose a basis for the hol. 1-forms on a general surface we use these to construct a map given by integration to a higher dim torus called the Jacobian.

Extra:

What happens at the vertex? Here we have a chart of the form

$$\phi(z) = z^{1/3}$$

where the exponent $1/3$ is chosen so that the cone angle of 6π maps to the cone angle of 2π. Let ψ be the inverse chart

$\psi(z) = z^3$. $\quad \psi^*(dz) = \frac{d\psi}{dz} \cdot dz = 3z^2 dz$.

Thus we see that the natural 1-form has a zero of order 2 at the vertex.

In general a zero of order k for the 1-form corresponds to a cone angle of $2\pi(k+1)$.