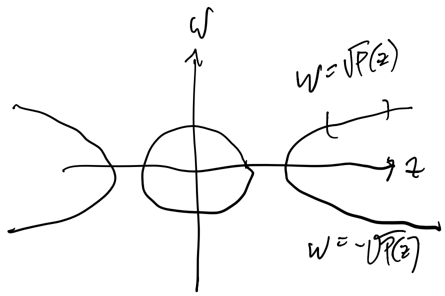
How do hyper-elliptic surfaces arise historically?

$\int \frac{dz}{\sqrt{P(z)}}$

Working on $V_p = \{(z,w) : w^2 = P(z)\}$ this becomes a well defined expression

$\int \frac{dz}{w}$ on V_p.



This becomes an integral on the Riemann surface V_p.
What does this mean?

deg P	genus of V_p		integrals
1	0	sphere	inverse trig.
2	0		
3	1	torus	elliptic
4	1		
5	2	higher genus	hyper-elliptic
6	2		

"space" of surface is relevant

We will look at the torus case in detail.

Hol. 1-forms.

The fundamental difference between doing analysis in $\mathbb{C}$ and on a Riemann surface R is that instead of having one coordinate we have multiple coordinate systems.

In order to define an object in a Riemann surface we need to know

① how to express it in coordinates

and

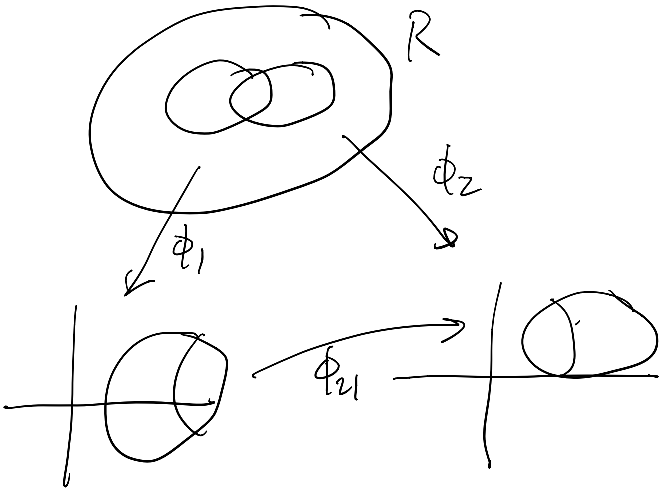
② how the expression changes if we change coordinates (using overlap functions).

Think about our discussion of valence or the property of being meromorphic. These properties do not change when we change coordinates.

In the case of integration there is a "change of variables formula" for the integral which means we have to be a little more careful.

In particular we introduce 1-forms which give us a coordinate independent way of talking about integration.

The idea of a 1-form is to look at the expression for an integral $\int f(z)dz$ and assign an independent existence to the expression $f(z)dz$.



$$\int_\gamma f(\phi(w))\frac{d\phi}{dz}dz = \int_{\phi(\gamma)} f(u)\,du$$

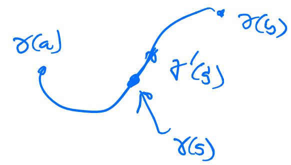
Push forward the path
pull back the integrand
with an extra factor $\frac{d\phi}{dz}$.

We want an integrand that makes
sense in a coordinate independent way.

$$\int \underbrace{f(z)dz}$$

Want to give independent
meaning to this expression.

What is the role of the $f(z)dz$ in path integration?



$$\int_\gamma \underbrace{f(z)dz} = \int_a^b \underbrace{f(\gamma(s))\cdot\gamma'(s)ds}$$

complex valued 1-form

this is a recipe for assigning a complex number to a tangent vector

$$\gamma'(s) \longmapsto f(\gamma(s))\cdot\gamma'(s)$$

viewed as a tangent vector

complex multiplication (complex linear)

viewed as a complex #.

Role of dz is to convert a tangent vector into a complex number.

In terms of real coords. $\gamma' = \begin{pmatrix}\dot x\\ \dot y\end{pmatrix}$

$$dx(\gamma') = \dot x$$

$$dy(\gamma') = \dot y$$

$$dz(\gamma') = \dot x + i\dot y = dx + i\,dy\,(\gamma').$$

for $dz = dx + i\,dy$.

Definition. A complex valued 1-form (say on $\mathbb{C}^n$) is a function that assigns a complex number to a vector and is complex linear on the set of vectors based at p.

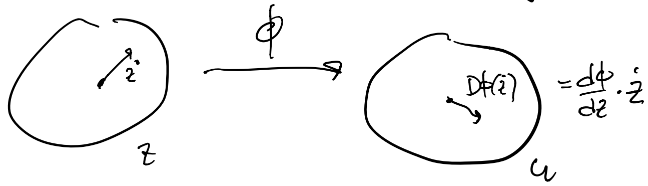
In $\mathbb{C}$ we can write it as $f(z)\,dz$ where f is a complex valued function.

How do they transform?

If ϕ is a map and θ is a 1-form then we define $\phi^*(\theta)(V_p) = \theta(D\phi_p(V))$ so

1-forms pull back.

In coordinates: say $\theta = g(u)\,du$



$\phi(z) = u$

$$\phi^*\big(g(u)\,du\big)(\dot{z}) = g(\phi(z))\,du\big(D\phi(\dot{z})\big) = g(\phi(u)) \cdot \left(\frac{d\phi}{dz} \cdot \dot{z}\right)$$

$$= g(\phi(u)) \cdot \frac{d\phi}{dz}\, dz(\dot{z})$$

So $\phi^*(g(u)\,du) = g(\phi(z)) \cdot \frac{d\phi}{dz} \cdot dz$.

(Evaluate $g(u)\,du$ at the image point $u = \phi(z)$).

One form builds in the factor $\frac{d\phi}{dz}$ that appears in the change of variables formula for the path integral.

Path integrals of 1-forms are defined in a coordinate independent way.

Given a 1-form $f(z)\,dz$ we can define a path integral $\int_\gamma f(z)\,dz$. Depends only on the 1 form.

Interpret $\gamma'(s)$ as a tangent a tangent vector based at the point $\gamma(s)$. The expression $f(z)\,dz$ takes a tangent vector and returns a complex number $f(\gamma(s))\cdot\gamma'(s)$ where γ' is the complex coordinate of the vector γ'.

Can think of $\gamma'(s)$ as a geometric object or a complex number.

Is a function which takes the geometric vector and returns the complex number.

(On a Riemann surface we can think of a geometric vector as something which only gets assigned to a number once a chart has been chosen.)

If we think of this function on tangent

vectors based at a particular point it is complex linear since it is given by complex mult.

Example. A complex valued 1-form in $\mathbb{C}^2$. Say we use z, w for our coordinate functions $\mathbb{C}^2 = \{(z,w) : z,w \in \mathbb{C}\}$.

Write vectors as column vectors then

$$dz\begin{pmatrix}\dot{z}\\ \dot{w}\end{pmatrix} = \dot{z}$$

$$dw\begin{pmatrix}\dot{z}\\ \dot{w}\end{pmatrix} = \dot{w}.$$