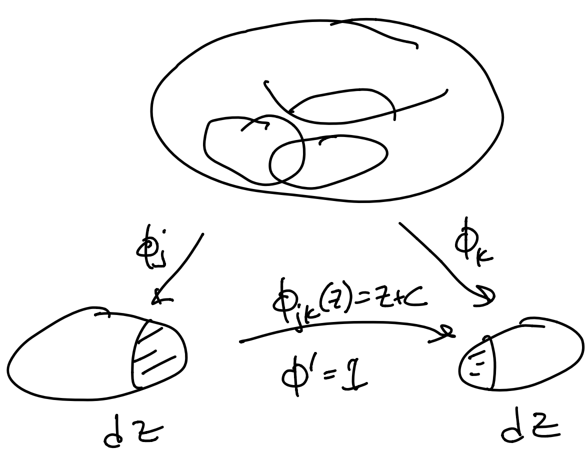
Holomorphic 1-forms played a large role in our discussion of surfaces of genus 1.
What happens in higher genus?

Recall that a simple way of constructing a 1-form on T^2 is to start with an atlas of charts where all transition functions are translations and take the $f_i = 1$.

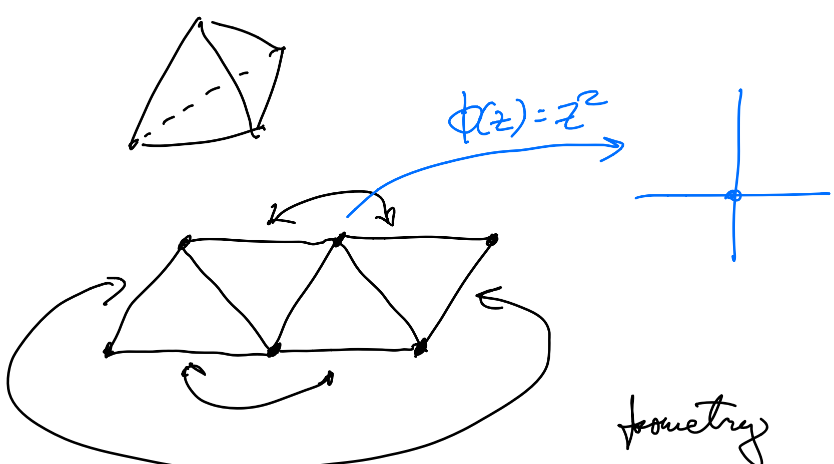


$\phi^*(dz) = dz.$

$$\phi^*(dz) = d(\phi^*(z)) = \frac{d}{dz}\phi \cdot dz = \frac{d(z+c)}{dz} dz = dz.$$

We can use a similar construction of holomorphic 1-forms in higher genus.

Recall that when we constructed a Riemann surface structure on the boundary of a polyhedron we used two types of charts one type for the vertices and



Geometry charts.

Vertex charts.

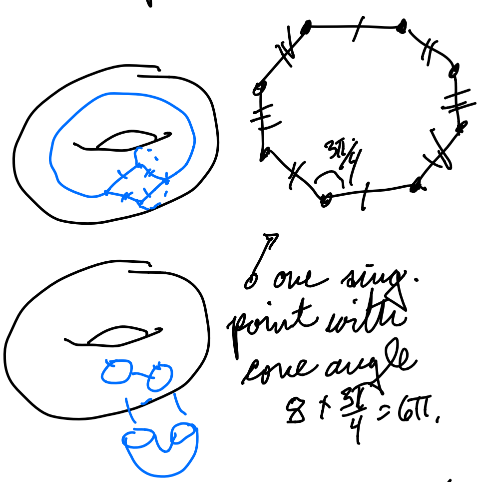
one type for every other point.

The overlaps for the second charts had the form $z \mapsto e^{i\theta}\cdot z + c$. As we saw in the construction of the pillowcase this construction works more generally whenever we have polygons in $\mathbb{C}$ glued together by isometries along the edges.

If we have a collection of polygons where all gluing maps have the form $z \mapsto z+c$ then we can build a hol. 1-form (at least away from the vertices).

Consider:

Example:



This construction gives a surface of genus 2 with a holomorphic 1-form with a zero of order 2.

$\boxed{\text{This is } \underline{\text{not}} \text{ a fund. domain for a lattice action. 1-forms in } g>1 \text{ } \underline{\text{not}} \text{ connected to } \Lambda\subset\mathbb{C}.}$

Here the isometries are all translations.

What happens at the vertex? Here we have a chart of the form

$$\phi(z) = z^{1/3}$$

where the exponent $1/3$ is chosen so that the cone angle of 6π maps to the cone angle of 2π. Let ψ be the inverse chart

$$\psi(z) = z^3. \qquad \psi^*(dz) = \frac{d\psi}{dz} \cdot dz = 3z^2 dz.$$

Thus we see that the natural 1-form has a zero of order 2 at the vertex.

In general a zero of order k for the 1-form corresponds to a cone angle of $2\pi(k+1)$.

In fact *every* hol 1-form arises from construction 1.

Given a hol 1-form θ on R remove from R the zeros and poles of θ.

Any $p \in R' = R - \{\text{zeros, poles}\}$. Let U be a simply connected set containing p.

Since θ ($= f(z)dz$) is closed and U is simply connected $\theta = dF$ in U'. $\phi(q) = \int_p^q \theta$

Now $F'(p) = f(p) \neq 0$

so F is locally invertible. On some smaller nbd $U_p \subset U'_p$ of p, F is a holomorphic bijection.

Let $\phi_p = F|U'_p$.

Let $\mathcal{A}$ consist of sets U_p and charts ϕ_p.

On an overlap ϕ_j and ϕ_k both satisfy $\phi' = \theta$ so $(\phi_j - \phi_k)' = 0$ and

$\phi_j - \phi_k = c$, $\quad \phi_j(z) = \phi_k(z) + c$. (3)

c is the constant of integration.

$$\phi_{jk}(z) = z + c.$$

If we pull back the form dz on $V_p \subset \mathbb{C}$ we get $\quad \phi^*(dz) = d\phi = \theta$.

At a zero ϕ is locally a branched cover $\phi(z) = z^n$ (in a good local coord. chart)

$\theta = d\phi = nz^{n-1}dz$ so order of the zero is $n-1$.

Lift the flat structure to get a cone point with cone angle $2\pi n$.

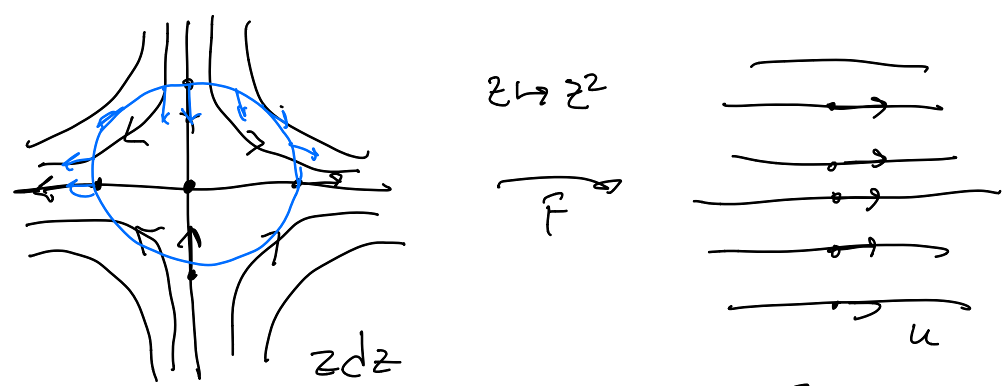
(Curvature is 2π − cone angle $= -2\pi$(order of zero))

Prop. If θ is a meromorphic 1-form on a compact Riemann surface R then

$\chi(R)$ = # poles of θ − # zeros of θ where both are counted with multiplicity.

Proof. Using the atlas we can construct a vector field on $\mathbb{R}'$.

We look at the vector field which in each translation chart is given by ∂_x.
Overlap functions preserve this vector field.
This vector



$u = \frac{z^2}{2}$
$du = 2z\,dz$

Zero of order 1 gives winding # -1.

$\theta = z\,dz \quad F(z) = \frac{z^2}{2}$.

In general the index of the singular point corresponding to $z^n dz$ is $-n$.

Want a vector $\dot{z}$ at $z \in S^1$ mapping to 1.
Take $\dot{z} = z^{-4}$. $\qquad DF_z(\dot{z}) = z^4 \cdot z^{-4} = 1$.

winding # -4.

④

According to the Poincaré index thm.
$\chi(R)$ is the sum of the indices at the zeros: $\chi(R) = \sum_{p \in \Sigma} -$ order of zero at p.

Example. A holomorphic 1-form on a surface of genus 1 has no zeros.

A holomorphic 1-form on a surface of genus greater than 1 <u>must</u> have a zero since $g > 1 \Rightarrow \chi < 0$.

Another new feature in genus 2 is that a given surface R has a 2 dimensional space of holomorphic 1-forms so the connection between conformal structures and hol. 1-forms is not so immediate.

If we fix R we can find 1-forms θ, θ' with zeros at different points on R. This implies that θ' is not a multiple of θ.

The collection of 1-forms on R is still an important invariant. We can consider the collection of homomorphisms

$$\pi_1(R) \longrightarrow \mathbb{C}$$

obtained by integrating hol. 1-forms. This is a subspace of dim g inside $\mathrm{Hom}(\pi_1(R), \mathbb{C})$ which is a $\mathbb{C}$ vector space of dim 2g.

(This discussion is most naturally phrased in terms of cohomology.)

This is an interesting invariant but it does not allow us to construct a moduli space as it did in $g=1$.

How might we construct a moduli space of genus 2 surfaces?

There is a second way to construct M_1. By means of the Weierstrass construction a surface of genus 1 is conformally equivalent to a 2 fold branched cover of $\mathbb{C}_\infty$ branched over 4 points. Two such surfaces are conformally equivalent if there is an automorphism of $\mathbb{C}_\infty$ taking one quadruple to the other.

It turns out that every surface of genus 2 arises from this hyper-elliptic construction as a branched cover over 6 pts in $\mathbb{C}_\infty$.

In the genus 1 case we identified the moduli space with a space of polynomials of degree 3. Perhaps this suggests that the moduli space has an algebro-geometric interpretation in general.

This is indeed the case. The moduli space is a subject of attention in alg. geom. The moduli space also plays a role in string theory where strings are Riemann surfaces. The moduli space is the "space of strings". The connection between moduli spaces and physics has also led to new tools in alg. geometry.