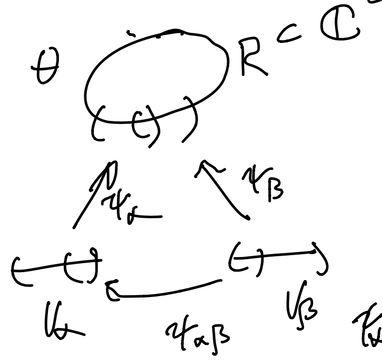
If $R \subset \mathbb{C}^2$ is a non-singular variety and θ is a 1-form on $\mathbb{C}^2$, $\iota : R \to \mathbb{C}^2$ is the inclusion then $\iota^*(\theta)$ is a well defined 1-form on R



$\psi_\alpha \circ \psi_{\alpha\beta} = \psi_\beta$.

in that given a chart $\mathcal{A}$ on R we get a family of 1-forms $\theta_\alpha = \psi_\alpha^*(\theta)$ satisfying $\psi_{\alpha\beta}^*(\theta_\alpha) = \theta_\beta$ since

$$\psi_{\alpha\beta}^*(\theta_\alpha) = \psi_{\alpha\beta}^* \circ \psi_\alpha^*(\theta) = (\psi_\alpha \circ \psi_{\alpha\beta})^*(\theta) = \psi_\beta^*(\theta) = \theta_\beta .$$

Definition. A complex valued function on $\mathbb{C}^2 = \{(z,w)\}$ is <u>holomorphic</u> if it can locally be given by a convergent power series $\sum_{j,k} a_{jk} z^j w^k$.

A function is <u>meromorphic</u> if it is a ratio of holomorphic functions.

Let $\pi_1 : \mathbb{C}^2 \to \mathbb{C}$ map $(z,w) \mapsto z$, $\pi_2((z,w)) = w$.

We write dz for $d\pi_1$ and dw for $d\pi_2$.

If $\mathbb{C}^2 = \{(x,y,u,v) : x,y,u,v \in \mathbb{R}\}$ then

$dz = dx + i\,dy \qquad dw = du + i\,dv.$

θ is a meromorphic 1-form on $\mathbb{C}^2$ if

$$\theta = f(z,w)\,dz + g(z,w)\,dw$$

with f, g meromorphic.

We get examples of meromorphic 1-forms on R by restricting meromorphic 1-forms on $\mathbb{C}^2$ to R.

$$
\begin{tikzcd}
R \subset \mathbb{C}^2 \arrow[dr, "\pi_z"] & \\
 & \mathbb{C} \\
V_\alpha \subset \mathbb{C} \arrow[uu, "\psi_\alpha"] \arrow[ur, "\pi_z \circ \psi_\alpha"'] &
\end{tikzcd}
$$

$u \in V_\alpha$, $\quad \pi_z \circ \psi_\alpha = \psi_1$

Write:

$$\psi_\alpha(u) = (\psi_1(u), \psi_2(u))$$

Then

$$\psi_\alpha^*(dz) = d(\pi_z \circ \psi_\alpha) = d(\psi_1) = \underbrace{\frac{d\psi_1}{du}}_{\text{holomorphic}} \cdot du.$$

$$\psi_\alpha^*\big(f(z,w)\cdot dz\big) = f(\psi_1(u), \psi_2(u)) \cdot \frac{d\psi_1}{du} \cdot du$$

In general

$$\psi_\alpha^*(f\,dz + g\,dw) = f \circ \psi_\alpha \frac{\partial \psi_\alpha}{\partial z} \cdot dz + g \circ \psi_\alpha \frac{\partial \psi_\alpha}{\partial w} dw,$$

meromorphic since all the fcns. are merom.

(If we write $z = \psi_1$, $w = \psi_2$ this looks like

$$f(u,v) \cdot \frac{dz}{du} dz$$

which is easier to remember.)

Consider the integral $\int \frac{dz}{\sqrt{1-z^2}}$ in the complex setting. In the real case $\int \frac{dx}{\sqrt{1-x^2}}$ we would identify this as arccos and define it for $x \in [-1,1]$ after choosing a principle branch of $\cos^{-1}$ once and for all.

In the complex setting this expression has meaning in $\mathbb{C} - \{-1, 1\}$ if we can identify which branch of the square root we are talking about.

We introduced Riemann surfaces as a technique for dealing with branches of functions. We can use them here.

Write $w = \sqrt{1-z^2}$ then $w^2 = 1-z^2$ and this equation defines a hyperelliptic surface $R = \{(z,w) : w^2 = 1-z^2\}$.

Consider the 1-form $\theta = \frac{dz}{w}$ on $\mathbb{C}^2$.
Consider the restriction of θ to R (or pullback $R \xrightarrow{\iota} \mathbb{C}^2$ $\iota^*(\theta)$).

$\frac{dz}{w}$ is a <u>well defined</u> form R.

$\phi_1(u) = u,\ \phi_2(u) = \sqrt{1-u^2}$

If $\phi_\alpha(u) = (u, \sqrt{1-u^2})$ is a coord. chart for R coming from some explicit choice of a branch of the square root term what is

$$\phi_\alpha^*\left(\frac{dz}{w}\right) = \frac{1}{\phi_2(u)}\cdot\frac{d\phi_1}{du}\cdot du$$

$$= \frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{du}\cdot du$$

$$= \frac{du}{\sqrt{1-u^2}}.$$

This is our original integral written in terms of u.

Conclusion we get a well defined 1-form on R and a well defined integration problem on R.

The ambiguity of the original integral has been replaced by the issue of choosing charts but this choice depends only on the point in R we are considering.

Definition. A hyperelliptic integral is an integral on the complex plane of the form $\int \frac{dz}{\sqrt{P(z)}}$.

When $\deg P = 2$ this is related to complex inverse trig functions and is periodic.

When $\deg P = 3$ or 4 these integrals are called elliptic integrals since they arise in finding the arclength of an ellipse.

We will see that they lead to "elliptic" analogs of trig functions.

In general the analysis of the integral is closely related to the analysis of the corresponding Riemann surface $R = \{w^2 = P(z)\}$.

Note that $\deg P = 3$ & 4 is exactly the case when R is a torus.

The functions that arise will be

doubly periodic and this is connected to $\pi_1(T^2) = \mathbb{Z}^2$.

We will return to this later in the course.

Automorphisms of the disk.

Recall that every connected Riemann surface can be written as S/Γ where S is simply connected and Γ is acting by holomorphic automorphisms. Recall that we also claimed that every simply connected Riemann surface is S^2, $\mathbb{C}$ or the disk. We will now describe these automorphism groups.

We start with the disk.

Without loss of generality we can consider the unit disk in $\mathbb{C}$ which we write as Δ.

Schwartz's lemma. Suppose $f: \Delta \to \Delta$ is holomorphic and that $f(0)=0$. Then either

(1) $|f(z)| < |z|$ for every non-zero z in Δ or

(2) $f(z) = e^{i\theta} z$ for some real constant θ.

Proof. $f(z) = a_1 z + a_2 z^2 + \dots$
$= z(a_1 + a_2 z + \dots)$
$= z \cdot g(z)$ g holomorphic.

For $r<1$ we can apply the maximum principle to g on the disk $\{|z| \le r\}$ and obtain

$$|g(z)| \le \sup_{|w|=r} |g(w)| < \frac{1}{r} \text{ since}$$

$$1 > |f(z)| = |z \cdot g(z)| = |z| \cdot |g(z)|$$

$$\text{so } |g(z)| < \frac{1}{|z|} \leq \frac{1}{r}$$

Letting $r \to 1$ we get $|g(z)| \leq 1$ in Δ.

If $|g| = 1$ at some point of Δ then g is constant by the maximum principal and $g = e^{i\theta}$ so (2) holds. Otherwise $|g| < 1$ and (1) holds.

Theorem. The elements of $\mathrm{Aut}(\Delta)$ are precisely the Möbius transformations of the form

$$f(z) = \frac{az + \bar{c}}{cz + \bar{a}} \text{ with } |a|^2 - |c|^2 = 1.$$

Proof. Elements of this form form a group.

$$\begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix}\begin{pmatrix} b & \bar{d} \\ d & \bar{b} \end{pmatrix} = \begin{pmatrix} ab+d\bar{c} & a\bar{d}+\bar{c}\bar{b} \\ bc+\bar{a}d & c\bar{d}+\bar{a}\bar{b} \end{pmatrix}$$

$$\det = a\bar{a} - c\bar{c} = 1.$$

$$|f(z)|^2 = \frac{|az+\bar{c}|^2}{|cz+\bar{a}|^2} = \frac{(az+\bar{c})(\bar{a}\bar{z}+c)}{|cz+\bar{a}|^2}$$

$$= \frac{a\bar{a}z\bar{z} + \bar{a}\bar{c}\bar{z} + acz + c\bar{c}}{|cz+\bar{a}|^2}$$

$$1 - |f(z)|^2 =$$

$$\frac{c\bar{c}z\bar{z} + \cancel{\bar{a}\bar{c}\bar{z}} + \cancel{acz} + a\bar{a} - a\bar{a}z\bar{z} - \cancel{\bar{a}\bar{c}\bar{z}} - \cancel{acz} - c\bar{c}}{|cz+\bar{a}|^2}$$

$$= \frac{1 - (c\bar{c} - a\bar{a})z\bar{z}}{|cz+\bar{a}|^2} = (1+|z|^2)\frac{1}{|cz+\bar{a}|^2}$$

$$f'(0)$$