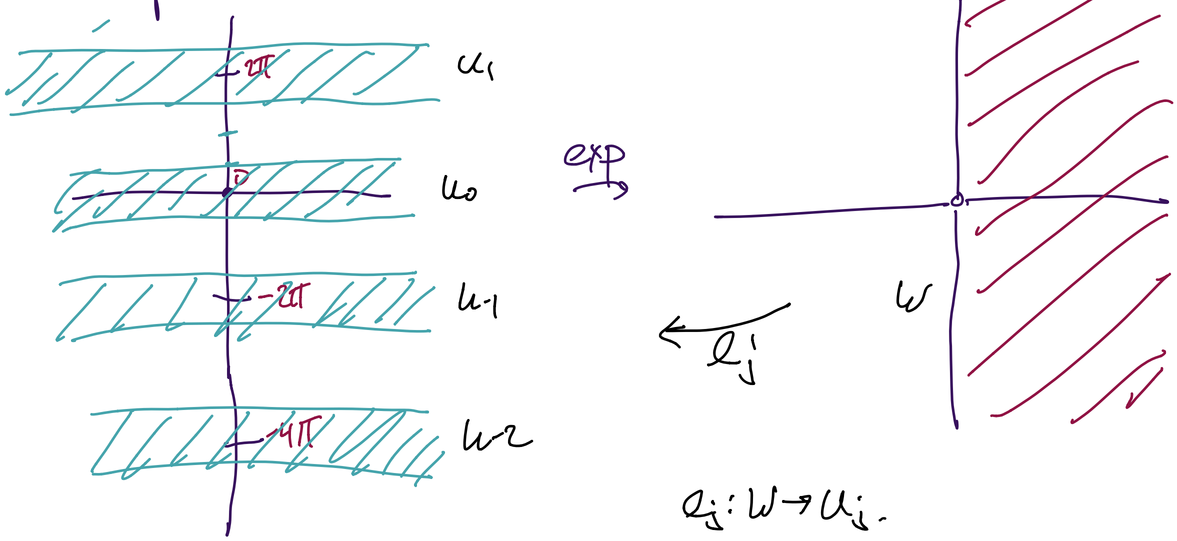
Recall that a map $F: X \to Y$ is a <u>covering map</u> if each point $p \in Y$ has a nbd. which is <u>evenly covered</u> and $U \subset Y$ is evenly covered if $F^{-1}(W)$ is a disjoint union of sets W_j where $F|U_j$ is a homeomorphism.

Example: $\exp: \mathbb{C} \to \mathbb{C} - \{0\}$. or $z \mapsto e^z$.



$\ell_j: W \to U_j$.

In this case we can think of the maps $\ell_j: W \to U_j$ as branches of the logarithm. We can also obtain these "branch" maps by integration of the 1 form $\frac{dz}{z}$.

$df = f'(z)dz$

Remark: $\exp^*\left(\frac{dz}{z}\right) = \frac{de^z}{e^z} = \frac{e^z dz}{e^z} = dz.$

We know that $\frac{dz}{z}$ does not have an anti-derivative on $\mathbb{C}-\{0\}$ but we know that restricted to any simply connected set $\frac{dz}{z}$ does have an anti-derivative and this anti-derivative is given by integrating along curves.

Further any two anti-derivatives differ by a constant.

$$l_j(w) = (2\pi i)\cdot j + \int_1^w \frac{dz}{z}.$$

Given a covering space $F: X \to Y$ we define the deck group Γ to consist of maps $g: X \to X$ so that $F \circ g = F$.

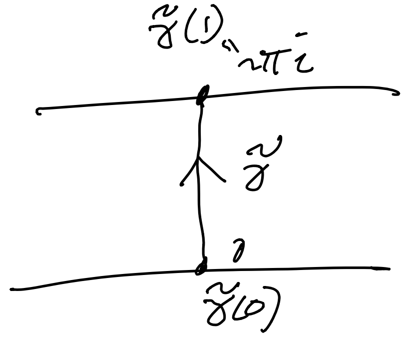
$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "F"'] & & X \arrow[dl, "F"] \\ & Y & \end{tikzcd}$$

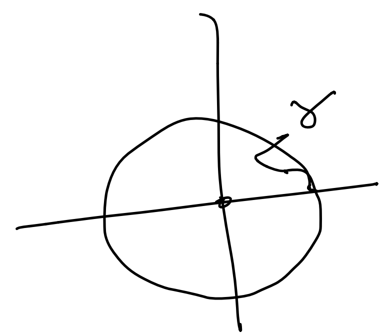
In our example Γ corresponds to $2\pi i \cdot \mathbb{Z}$ acting on $\mathbb{C}$ by translation. When X is simply connected we can identify Γ with $\pi_1(Y)$.

We do this by taking a loop downstairs $\gamma : [0,1] \to Y$ lifting upstairs $\tilde{\gamma} : [0,1] \to X$. We map γ to g where

$\gamma(\tilde{p}(0)) = \tilde{p}(1)$.

$$g(\tilde{\gamma}(0)) = \tilde{\gamma}(1).$$





In general for any U and holomorphic 1-form $f(z)dz$ on U path integration gives a group homomorphism $h: \pi_1(U) \to \mathbb{C}$ where $h(\gamma) = \int_\gamma f(z)dz$.

In this case we have:

$\boxed{\text{Problem. Check this.}}$

$$\underbrace{\pi_1(\mathbb{C} - \{0\})}_{\text{deck group}} \cong \mathbb{Z}$$ and we can realize an explicit identification by sending a loop γ in $\mathbb{C}-\{0\}$ to $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z}$.

Def. $\text{wind}(\rho, 0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z}$.

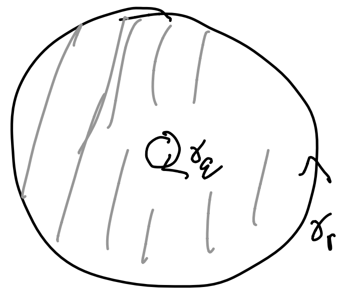
So $\text{wind}(\rho, 0)$ records the homotopy type of ρ as an integer. "How many times ρ maps around 0.

Def. $\text{wind}(\rho, z_0) = \frac{1}{2\pi i}\int \frac{dz}{z - z_0}$. "How many times ρ maps around z_0."

Cauchy Integral Formula.

(In 1-complex variable want a proof from first principles since it is this result that leads to power series representation.)

$$\int \frac{f(z)}{z}\,dz$$



f is holomorphic in $D-\{0\}$, $\frac{1}{z}$ is hol. in $D-\{0\}$. $\frac{f(z)}{z}$ is hol in $D-\{0\}$ so closed in $D-\{0\}$.

Gauss' thm.

$$0=\int_{U} d\left(\frac{f(z)}{z}\,dz\right) = \int_{\partial U} \frac{f(z)}{z}\,dz = \int_{\partial D_\varepsilon} \frac{f(z)}{z}\,dz - \int_{\partial D_r} \frac{f(z)}{z}\,dz.$$

$U = D_r - D_\varepsilon$

$$t \mapsto r\cos t + i\cdot r\sin t = \gamma(t) \qquad \gamma'(t) = -r(\sin t + i\cos t)$$

$$\int_{\partial D_r} \frac{f(z)}{z}\,dz = \int_0^{2\pi} \frac{f(\gamma(t))}{\gamma(t)}\cdot \gamma'(t)\,dt$$

=

Write $f(z) = f(0) + (f(z) - f(0))$

$\int_t O(r)\cdot r\,dt$

$= o(r) \to 0.$

$f(0)\cdot \mathrm{wind}(\gamma_\varepsilon, 0)$

$$= \frac{1}{2\pi i}\int_{\gamma_\varepsilon} \frac{f(0)}{z}\,dz + \int_{\gamma_\varepsilon} \frac{f(z)-f(0)}{z-0}\,dz$$

goes to 0 as $r \to 0$.

$n \cdot f(0)$

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} \to f'(0).$$

so this quotient is bounded for r small.

Cauchy formula gives a way to reproduce the values of f on a disk from the values of f on the boundary of the disk. leads to analyticity and uniform convergence

Morera's thm.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \text{ in } D.$$

integral formula for a_n.

Residue theorem:

$$\int f dz$$

Morera's Thm. f is holomorphic in a disk iff $\int_\gamma f\,dz = 0$ for any γ.

$\Rightarrow$ Uniform limit of holomorphic functions is holomorphic. (Problem)

Beardon p. 8

Prop. If f is holomorphic and non-constant on a connected domain U then at each $z \in U$ there is some first coefficient so

$$f(z) = f(z_0) + (z - z_0)^N \left(a_N + a_{N+1}(z - z_0) + \cdots\right)$$

with $a_N \neq 0$.

since a_j varies continuously with z

Proof. The set where $a_j = 0$ is closed so the set where all a_j vanish is closed. On the other hand the set where all a_j vanish is open by analyticity. Since this set is not all of U it is empty by connectivity.

given $c \in \mathbb{C}$

Corollary. The set of z in U with $f(z)=c$ is isolated.

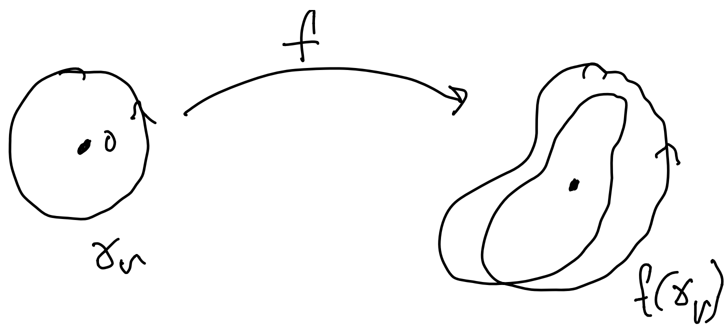
Proof. Say $f(z_0)=c$ then near z_0

$$f(z) = f(z_0) + (z-z_0)^n g(z) \quad \text{with } g(z_0)\neq 0.$$

so $f(z)-f(z_0) = (z-z_0)^n g(z)$. If $z \neq z_0$ is close to z then this difference is non-zero by the cont. of g.

Winding numbers and counting zeros of holomorphic functions.

Say f is holomorphic but not constant and $f(0)=0$. Since 0's are isolated we can find a disk D_r around 0 so that 0 is the only point in D_r mapping to 0.



In particular $f(\gamma_r) \subset \mathbb{C} - \{0\}$.

What is the winding number of $f(\gamma_r)$?

Now $f(z) = z^n \cdot g(z)$ with $g(0) \neq 0$.

$$\operatorname{wind}(f(\gamma_r), 0) = \frac{1}{2\pi i}\int_{f(\gamma_r)} \frac{dz}{z}$$

$$= \frac{1}{2\pi i}\int_{\gamma_r} f^*\left(\frac{dz}{z}\right)$$ naturality of path integration

$$= \frac{1}{2\pi i}\int_{\gamma_r} \frac{f'(z)\,dz}{f(z)}$$

$$= \frac{1}{2\pi i}\int_{\gamma_r} \frac{n z^{n-1} g(z) + z^n g'(z)}{z^n\, g(z)}$$

$$= \frac{1}{2\pi i}\int_{\gamma_r} \frac{n\,dz}{z} + \frac{g'(z)}{g(z)}\,dz$$

$$= n.$$

Let us define

n = valence of f at 0. Note that $n=1$ iff $f'(0) \neq 0$. (Define valence more generally?)

General formula.

Let $f: D \to \mathbb{C}$ be holomorphic

then $\frac{1}{2\pi i}\int \frac{f'(z)}{z}\,dz$ = winding # of $f(\partial D)$

$$= \sum_{z_j : f(z_j)=0} n_j$$

= # of solutions counted with multiplicity.