We can define

a complex 1-form is an expression $f(z)\,dz$ which

transforms by pulling back

$$F^*\big(g(u)\,du\big) = g\big(F(z)\big)\cdot\frac{dF}{dz}\cdot dz$$

$$\boxed{F^*(du) = \frac{dF}{dz}\cdot dz}$$

(Write $u = F(z)$ $\quad g(u)\cdot\frac{du}{dz}\,dz$)

There is a natural way to associate a 1-form to a function.

Say f is holomorphic, $\dot z$ is a vector based at z then the directional derivative of f in direction $\dot z$ is a complex number which depends complex linearly on $\dot z$ at z. We call this 1-form df.

So $df = \frac{df}{dz}\,dz$

$$\boxed{df(\dot z) = \frac{df}{dz}\cdot\dot z = \frac{df}{dz}\cdot dz(\dot z)}$$

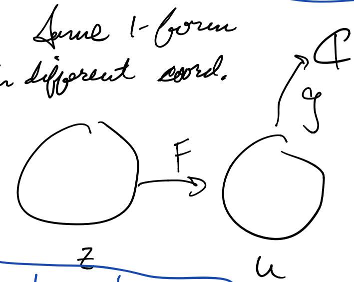
Special case: dz on $\mathbb{C}$ is the operator d applied to the function z. dz, dw are d applied to coord fns. z, w in $\mathbb{C}^2$.

Formula for exterior derivative:

$$dg(u) = \frac{dg}{du}\,du$$

$dg = \frac{dg}{dz}\cdot dz$. Same 1-form expressed in different coord. systems.

Version of pull back



The operator d is <u>natural</u>. Respects coordinate changes.

pulling back 1-form

pullback of function

Claim $F^*(dg) = d\,F^*(g) = d(g\circ F)$

Pullback without pulling back?

$$F^*\left(\frac{dg}{du}\cdot du\right) \qquad\qquad \frac{d}{dz}(g\circ F)\,dz$$

$$\frac{dg}{du}(F(z))\cdot du \qquad\qquad \frac{d}{du}g(F(z))\cdot\boxed{\underbrace{\frac{dF}{dz}\cdot dz}_{=\,du}}$$

(Easy to remember way to compute $F^*(dg)$.)

Those who are knowledgable about 1-forms know that they fit into a general theory of n-forms. Operator d takes an n-form to an n+1-form. 0-form is a function.

We say a 1-form θ is <u>closed</u> if $d\theta = 0$.
If a 1-form is closed then locally there is a function f with $df = \theta$. Exercise: If f is holomorphic then $f\cdot dz$ is <u>closed</u>. This follows from

the Cauchy-Riemann equations.

Essentially equivalent to the fact that anti-derivatives exist locally. Anti-derivative: F is the anti-derivative of f if $F' = f$. New language $dF = f\,dz$. ← 1-form

Local existence of an anti-derivative for ← function a holomorphic function is essentially equivalent to hol. 1-forms being closed.

$d^2F = d(f\,dz) = 0$ so $f\,dz$ is closed.

In particular this explains why holomorphic 1-forms are particularly nice examples of complex valued 1-forms.

Holomorphic 1-forms also called "holomorphic differentials"* and "Abelian differentials" (ie connected to Abel).

For the next discussion it is useful to talk about an atlas of inverse charts.

This is an equivalent concept but sometimes it is more natural for what we want to do.

Say $\phi_\alpha : U_\alpha^{\subset R} \longrightarrow V_\alpha^{\subset R}$ is a chart in $\mathcal{A}$ where $\alpha \in A$.

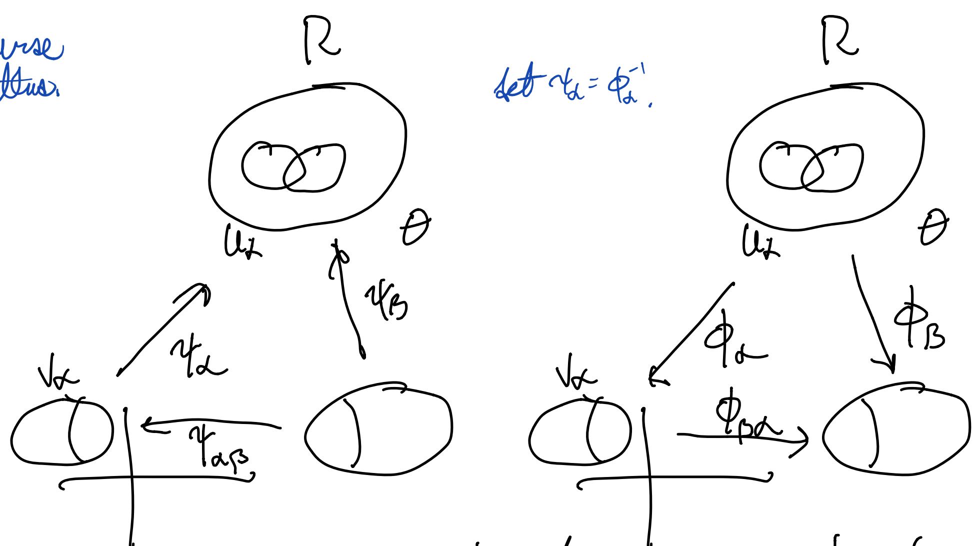
Let $\mathcal{A}^{-1}$ consist of "inverse atlas"

Let $\psi_\alpha : V_\alpha \longrightarrow U_\alpha$ be inverse charts $\psi_\alpha = \phi_\alpha^{-1}$

Inverse atlas.

Let $\psi_\alpha = \phi_\alpha^{-1}$.

Atlas.



$\psi_\alpha \circ \psi_{\alpha\beta} = \psi_\beta$ where defined.

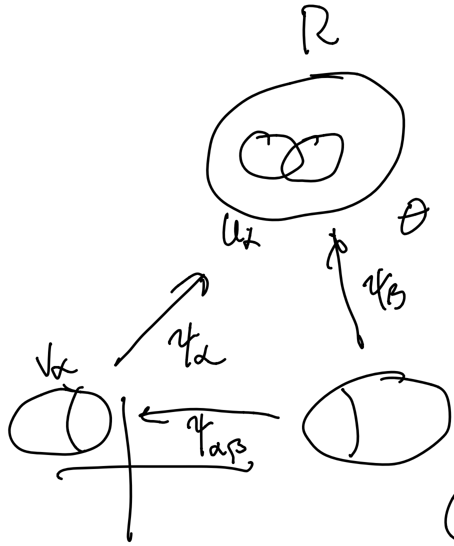
$\phi_{\beta\alpha} \circ \phi_\alpha = \phi_\beta$

Let $\psi_{\alpha\beta} = \phi_{\beta\alpha}^{-1}$.

How do we define forms on surfaces?

Say R is a surface (not nec. a Riemann surface) and $\mathcal{A}$ is an atlas for R.

If there was a 1-form θ on R then for each inverse chart ψ_α there would be a 1-form $\theta_\alpha = \psi_\alpha^*(\theta)$ on V_α. Since



$$\psi_\alpha \circ \psi_{\alpha\beta} = \psi_\beta$$

it would be the case that

$$(\psi_\alpha \circ \psi_{\alpha\beta})^* = \psi_\beta^*(\theta) \text{ but}$$

$$(\psi_\alpha \circ \psi_{\alpha\beta})^* = \psi_{\alpha\beta}^*\left(\psi_\alpha^*(\theta)\right) \text{ so}$$

$$\psi^*_{\alpha\beta}(\psi_\alpha^*(\theta)) = \psi_\beta^*(\theta) \text{ or}$$

$$\psi^*_{\alpha\beta}(\theta_\alpha) = \theta_\beta$$

(where this makes sense).

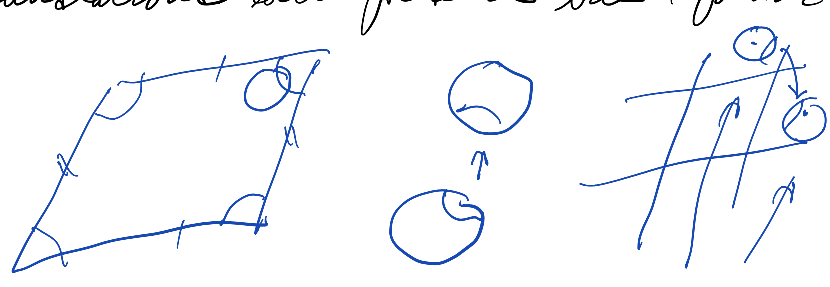
If we wanted to do something with this 1-form like integrate over a short path then we could integrate in a coordinate chart and the particular choice of chart would not affect the answer.

How do you integrate over a long path?

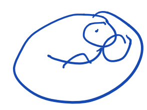
Example: The torus.

$T = \mathbb{C}/\Lambda$. The action of Λ preserves the holomorphic 1-form dz.

In general any function where the coordinate change maps are given by translations will preserve the 1-form dz.



Geometric theory of hol. 1-forms.



Example. Let $f: V \longrightarrow \mathbb{C}\times\mathbb{C}$ be the inclusion of the hyper-elliptic surface. (z,w) coordinates. dz is a hol. 1-form on $\mathbb{C}\times\mathbb{C}$. $\frac{dz}{w}$ is a meromorphic 1-form. $f^*\left(\frac{dz}{w}\right)$ is a meromorphic 1-form on V.

Alternatively we can pull back the functions (or restrict the functions) π_z and π_w to functions z, w on V. Then dz is the exterior derivative of the function $z^{\text{on } V}$. (Note that unlike dz in $\mathbb{C}$, dz on V may have developed some zeros.)