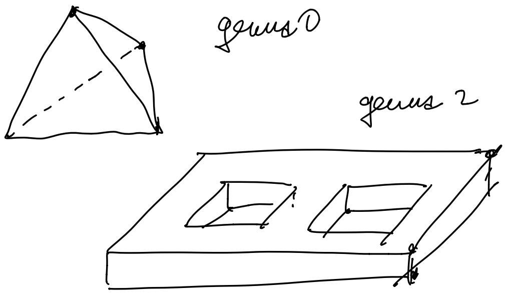


Polyhedral surface.

Polyhedron in $\mathbb{R}^3$ is a topological surface consisting of faces, edges and vertices.

We will show that a polyhedron determines a Riemann surface. This construction gives a large number of examples (unlike previous constructions) including examples of Riemann surface structures on oriented surfaces of every genus. These constructions each have a finite # of parameters that can be adjusted.

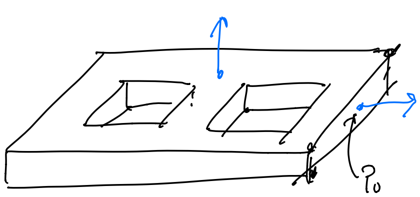
Step 1. We will start by constructing charts at interior points of faces.

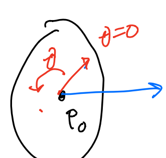
Our surface comes with an outward pointing unit normal.

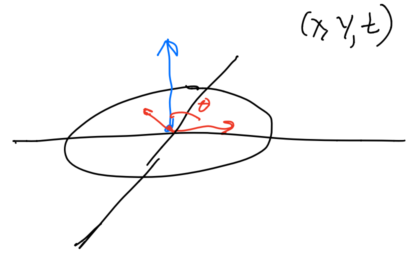
$x+iy \mapsto (x,y,0)$

$\mathbb{C} \to \mathbb{R}^3$

(x,y,t)



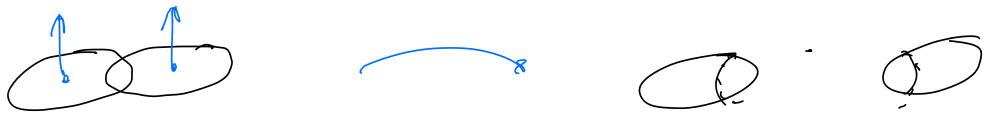




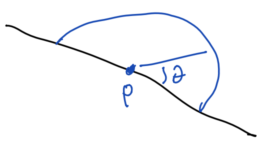
Polyhedral surface.

Construct polar coordinates in a nbd. of p. $0 \le r \le \delta_0$ $0 \le \theta \le 2\pi$.

If two points are in the same face then disks can overlap and transition functions have the form $z \mapsto z+c$



Step 2, Coordinates on edges



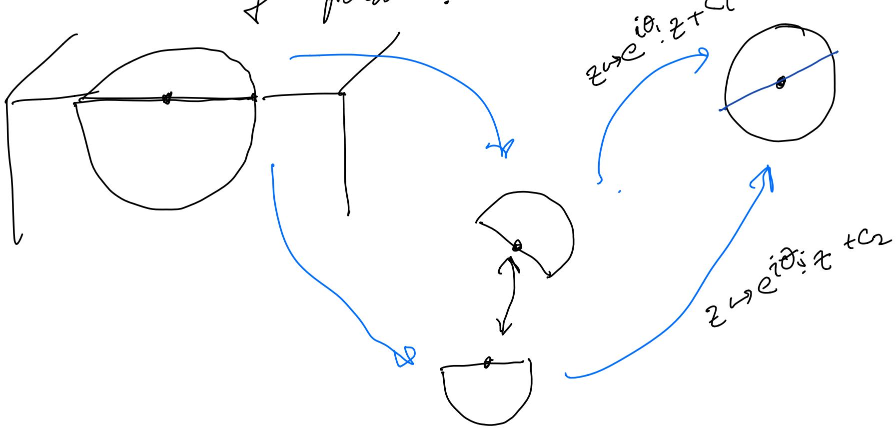
At a point p on the boundary of a face f_j we can construct a half-disk coordinate.

$0 \le r \le s_p$

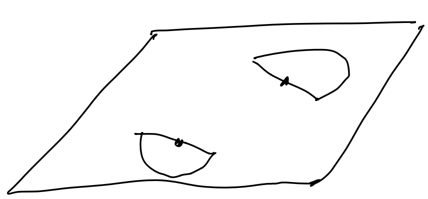
$0 \le \theta \le \pi.$

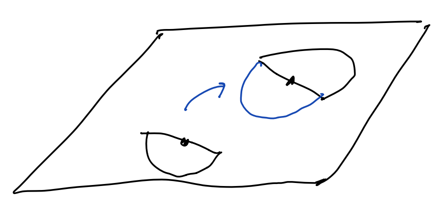
If we "put together" a pair of these maps we get a disk coordinate.

Edge points.



Decide which half-disk becomes the upper half and which the lower half.





At this point we have an atlas for P-vertices where all of the transition maps have the form $\phi_{jk}(z) \mapsto e^{\theta_{jk}} z + c_{jk}$. This is a holomorphic atlas of a special form and we will see this again.

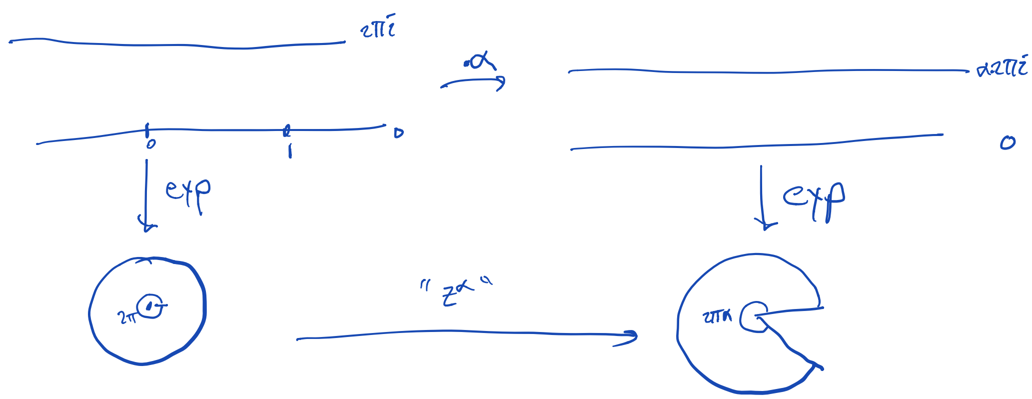
Step 3. Coordinates at vertices.

Remarks about the function $z \mapsto z^\alpha$.

We can make sense of this by writing

$z^\alpha = \exp(\log z^\alpha) = \exp(\alpha \log z)$.

We interpret this by choosing a branch of the logarithm. Now lets assume that α is real and positive. We get the following picture:

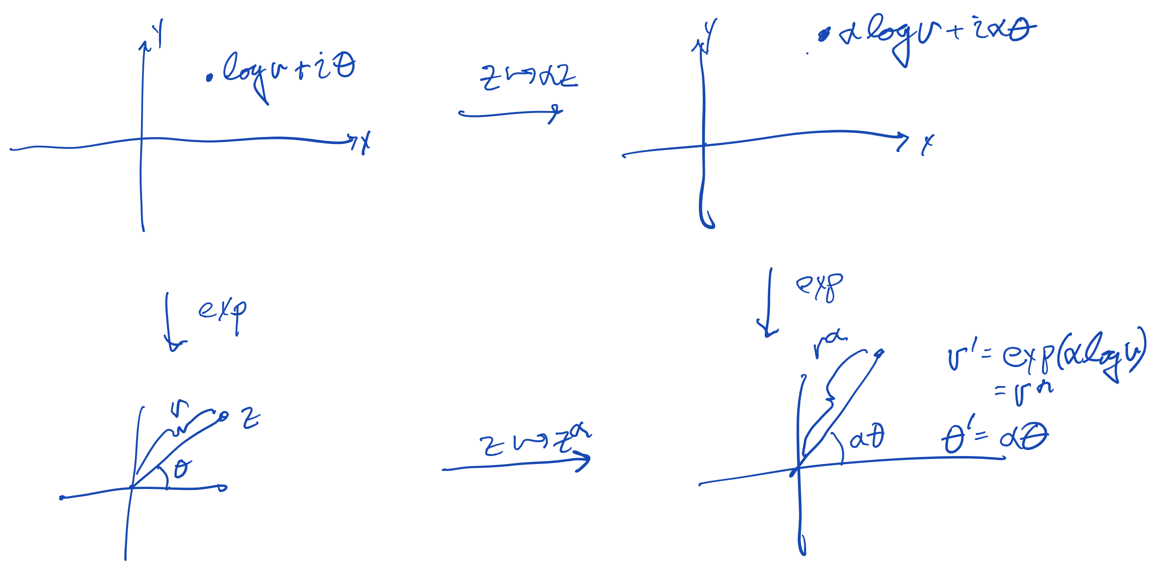


Choosing a different branch of the logarithm changes the lift of the map by a multiple of $2\pi i\alpha$ and changes the map by a rotation of $\exp(2\pi i\alpha)$.

Note that choosing x and y for our coordinates upstairs corresponds to using polar coordinates $r = e^x$ $\theta = e^{iy}$ downstairs

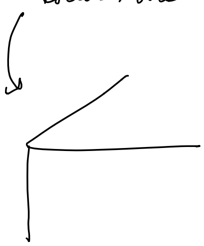
$$\exp(x+iy) = e^x \cdot e^{iy} = r\cdot\theta.$$

In polar coordinates $z \mapsto z^\alpha$ becomes $(r,\theta) \mapsto (r^\alpha, \alpha\theta)$.



(r,θ) coordinates
$0 \le r \le \delta_p$
$\theta_0 \le \theta \le \theta_1$

total cone angle $= \frac{3\pi}{4}$



Angle smaller than π.

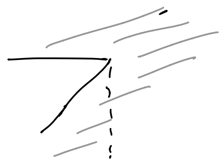
$\frac{3\pi}{2} \cdot \alpha = 2\pi$

$\alpha = \frac{4}{3}$

$z \mapsto z^{4/3}$

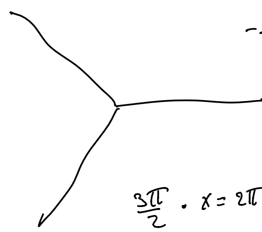
$-\frac{\pi}{2} \le \theta \le \pi$

$-\frac{3\pi}{4} \le \theta \le \frac{3\pi}{2}$



$(r,\theta) \mapsto (r^\alpha, \alpha\theta + \text{const.})$

$z \mapsto z^\alpha$



$\alpha = 2\pi \cdot \frac{2}{3\pi}$
$= \frac{4}{3}\pi$

$\frac{3\pi}{2} \cdot \alpha = 2\pi$

Adding in charts of this form give us transition functions of the form $z \mapsto \lambda z^\beta + c$ where z^β is a branch of the power function.

3 points.

① There is a Riemann surface atlas $\mathcal{A}$ for P (according to the strict definition of what a Riemann surface atlas is.

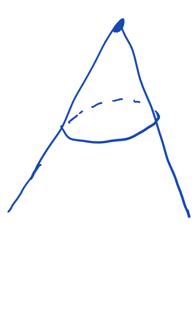
This definition creates a smooth surface (with a tangent bundle) which is homeomorphic to P.

② Our intuition does not exactly agree with the strict definition. We would like some of the structure of P beyond just the topology to be reflected in $\mathcal{C}$. In particular where P has a recognizable conformal structure we would like this to agree with the conformal structure given by $\mathcal{C}$.

③ Claim that any two atlases for P which are conformal away from the vertices are equivalent.

This says that even though P is not smooth itself, the conformal structure on P - {vertices} determines a <u>unique</u> conformal structure on all of P.

Simplest example: cone in $\mathbb{R}^3$ with



cone angle α.

Say we have

Recall the theorem about isolated singularities.

3 cases:

Removable singularity

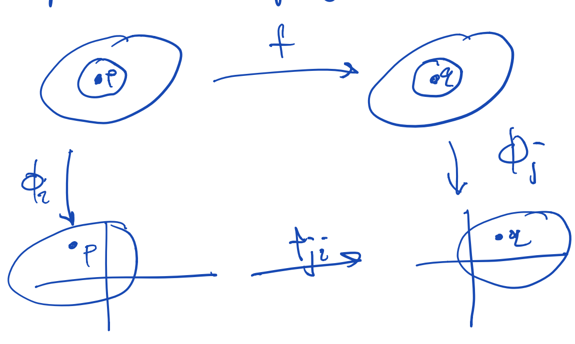
Pole

Essential singularity.

Theorem. (Removable singularities for atlases.)

Let R and R' be Riemann surfaces. Let Σ and Σ' be discrete subsets of R and R' and let $f: R \to R'$ be a homeomorphism taking Σ to Σ' which is a holomorphic map from $R - \Sigma$ to $R - \Sigma'$ then f is in fact a holomorphic map from R to R'.

Proof.



Need to check that f_{ji} is holomorphic.

f_{ji} is a homeomorphism which is holomorphic away from an isolated singular point.

It follows from the classical removeable singularities theorem that f is holomorphic at p.