Schwartz's lemma. Suppose $f:\Delta\to\Delta$ is holomorphic and that $f(0)=0$. Then either

(1) $|f(z)|<|z|$ for every non-zero z in Δ or

(2) $f(z)=e^{i\theta}z$ for some real constant θ.

In addition we have $|f'(0)|\le 1$. If equality holds we are in case (1) above otherwise we are in case 2.

Proof. $f(z)=a_1z+a_2z^2+\dots$
$$=z(a_1+a_2z+\dots)$$
$$=z\cdot g(z)\quad g \text{ holomorphic}$$

For $r<1$ we can apply the maximum principle to g on the disk $D_r = \{|z| \le r\}$ and obtain

$$|g(z)| \le \sup_{|w|=r} |g(w)| < \frac{1}{r} \qquad (*)$$

since on D_r:

$$1 > |f(z)| = |z \cdot g(z)| = |z| \cdot |g(z)|$$
$$= r \cdot |g(z)|$$

since max occurs on ∂D_r by the maximum principle.

(Note that there are two versions of the maximum principle. Here we are using the version which says that a cont. fcn. on the closed disk which is holomorphic

in the interior takes its maximum on the boundary. Below we use a refined version which implies that if a holomorphic function on the open disk achieves its maximum then it is constant.)

Letting $r \to 1$ in equation * we get

$|g(z)| \le 1$ in Δ. Note in particular that
$|g(0)| = |f'(0)| \le 1$.

If $|g| = 1$ at some point of the open disk Δ then by the second version of the maximum principle g is constant $g(z) = c$. Thus $|c| = 1$ and $c = e^{i\gamma}$.

g is constant by the refined maximum principal and $g(z) = e^{i\theta} \cdot z$ so (2) holds. Otherwise $|g| < 1$ and (1) holds.

To prove the last two statements note that g is defined and holomorphic on Δ and since $f(z) = z \cdot g(z)$ we have $f'(0) = g(0)$.

If $|f'(0)| = 1$ then $|g|$ achieves its maximum in Δ and we are in case 2. If $|f'(0)| < 1$ then $|g(0)| < 1$ then $|g(z)|$ cannot be 1 at any $z \in \Delta$ so we are in case 1.

Theorem. The elements of $\mathrm{Aut}(\Delta)$ are precisely the Möbius transformations of the form

$$f(z) = \frac{az + \bar{c}}{cz + \bar{a}} \quad \text{with } |a|^2 - |c|^2 = 1.$$

Proof. Elements of this form form a group.

Elements are closed under composition:

Write $\mathbb{C}_\infty$ as $\mathbb{CP}^1 = \mathbb{C}^2/\mathbb{C}^*$ $PGL(2,\mathbb{C})$ acts on $\mathbb{CP}^1$. map from matrix to LFT

$$\begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix}\begin{pmatrix} b & \bar{d} \\ d & \bar{b} \end{pmatrix} = \begin{pmatrix} ab+d\bar{c} & a\bar{d}+\bar{c}\bar{b} \\ bc+\bar{a}d & c\bar{d}+\bar{a}\bar{b} \end{pmatrix}$$

So conjugation condition holds.

$$\det = a\bar{a} - c\bar{c} = |a|^2 - |c|^2 = 1.$$

Mult. of det means cond $|a|^2-|c|^2=1$ is preserved

Closed under taking inverses:

$$\begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix}^{-1} = \begin{pmatrix} 1/a & -\bar{c} \\ -c & 1/\bar{a} \end{pmatrix}$$

has same form. Det condition holds.

Check that the disk is taken into the disk.

$$|f(z)|^2 = \frac{|az+\bar c|^2}{|cz+\bar a|^2} = \frac{(az+\bar c)(\bar a\bar z + c)}{|cz+\bar a|^2}$$

$$= \frac{a\bar a z\bar z + \bar a\bar c\bar z + acz + c\bar c}{|cz+\bar a|^2}$$

$$1-|f(z)|^2 =$$

$$\frac{c\bar c z\bar z + \cancel{\bar a\bar c\bar z} + \cancel{acz} + a\bar a - a\bar a z\bar z - \cancel{\bar a\bar c\bar z} - \cancel{acz} - c\bar c}{|cz+\bar a|^2}$$

$$= \frac{1-(c\bar c - a\bar a)z\bar z}{|cz+\bar a|^2} = (1-|z|^2)\frac{1}{|cz+\bar a|^2}$$

Thus $1-|f(z)|^2 = (1-|z|^2)\cdot\frac{1}{|cz+\bar a|^2}$

So if $z\in\Delta$ so $1-|z|^2<1$ then $1-|f(z)|^2<1$ and $f(z)\in\Delta$. QED

Same argument applies to f^{-1} so $f(\Delta)=\Delta$. QED.

Observe that:

$$f'(z) = \left(\frac{az+b}{cz+d}\right)' = \frac{a(cz+d)-c(az+b)}{(cz+d)^2}$$

$$= \frac{\cancel{acz}+ad-\cancel{acz}-bc}{(cz+d)^2}$$

$$= \frac{ad-bc}{(cz+d)^2}.$$

Applied to

$\begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix}$ we get $\dfrac{1}{(cz+d)^2}$

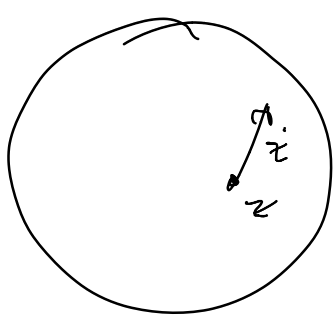
so $|f'(z)| = \dfrac{1-|z|^2}{1-|f(z)|^2}$.

$$\boxed{\text{Thus } |f'(0)| = \frac{1-|f(z)|^2}{1-|z|^2}.}$$

A consequence of these calculations is that a linear fractional transformation preserves the hyperbolic metric on Δ.

For $z \in \Delta$ let $\quad S(z) = \dfrac{2}{1-|z|^2}$.

Define a metric on Δ so that the length of $(z, \dot{z})$ is $S(z) \cdot |\dot{z}|$.



Claim. f preserves this metric.

Applying f to $(z, \dot{z})$ gives $(f(z), f'(z) \cdot \dot{z})$

Check $\quad \delta(z) \cdot |\dot{z}| \stackrel{?}{=} \delta(f(z)) \cdot |f'(z)| \cdot |\dot{z}|$

or $\quad |f'(z)| \stackrel{?}{=} \dfrac{\delta(z)}{\delta(f(z))}.$

$$\text{But } \frac{\delta(z)}{\delta(f(z))} = \frac{\dfrac{2}{1-|z|^2}}{\dfrac{2}{1-|f(z)|^2}} = \frac{1-|f(z)|^2}{1-|z|^2} \stackrel{!}{=} |f'(z)|.$$

Note that any choice of a constant gives an invariant metric. The 2 yields a metric with curvature 1.

Remark. For any $h: \Delta \to \Delta$ we can define $|h'(w)|_{hyp}$ to be

$$(w, \dot{w}) \longrightarrow (h(w), h'(w) \cdot \dot{w})$$

the ratio $|(h(w), h'(w)\cdot \dot{w})|_{hyp}$ to $|(w, \dot{w})|_{hyp}$.

This is the factor by which h changes hyp length at w.

$$|h'(w)|_{hyp} = \frac{\delta(h(w))\cdot |h'(w)|\cdot |\dot{w}|}{\delta(w)\cdot |\dot{w}|}$$

$$= |h'(w)|\cdot \frac{\delta(h(w))}{\delta(w)}.$$

Previous calculation says $|f'(w)|_{hyp} = 1$ if f is in G.

Define the length of a path γ to be

$L(\gamma) = \int_\gamma \delta(\gamma(t))\cdot |\gamma'(t)|\,dt$ and

$$\rho(z, w) = \inf_{\substack{\gamma(0)=z \\ \gamma(1)=w}} L(\gamma).$$

If $f \in G$ then $\rho(f(z), f(w)) = \rho(z, w)$.

Claim. G acts transitively on Δ.

Say $w \in \Delta$.

Find a, c with $|a|^2 - |c|^2 = 1$ with $f(0) = w$

or $\dfrac{a \cdot 0 + \bar{c}}{c \cdot 0 + \bar{a}} = \dfrac{\bar{c}}{\bar{a}} = w$.

Write $\bar{a} = r$ $\quad \bar{c} = r \cdot w$. Want to choose r so that $|a|^2 - |c|^2 = 1$. We have

$$|a|^2 - |c|^2 = r^2 - r^2 \cdot |w| = r^2(1 - |w|).$$

Set $r = \dfrac{1}{\sqrt{1 - |w|}}$. This is justified since $|w| < 1$ so $1 - |w| > 0$.

If w_0, w_1 are any two points in D choose f_0, f_1 with $f_j(0) = w_j$ so

$$f_1(f_0^{-1}(w_0)) = f_1(0) = w_1 \quad \text{and} \quad f_1 \circ f_0^{-1}$$

takes w_0 to w_1.

Thm. (Schwarz-Pick). Let h be any holomorphic map $h: \Delta \to \Delta$. Then h does not increase the hyperbolic distance.

Proof. Say $h(w_1) = w_2$

Say $f_1(0) = w_1$, $f_2(0) = w_2$.
Now $f_2^{-1} \circ h \circ f_1$ takes 0 to 0 and takes Δ to Δ. We can apply the standard Schwarz lemma to get

$$(f_2^{-1} \circ h \circ f_1)'(0) \leq 1.$$

Measuring the effect on the hyperbolic metric gives

$$|(f_2^{-1})'|_{hyp} \, |h'|_{hyp} \, |f_1'|_{hyp} \leq 1$$

so $|h'|_{hyp} \leq 1$.

$$f_2'(h f_1(0)) \cdot h'(f_1(0)) \cdot f_1'(0)$$

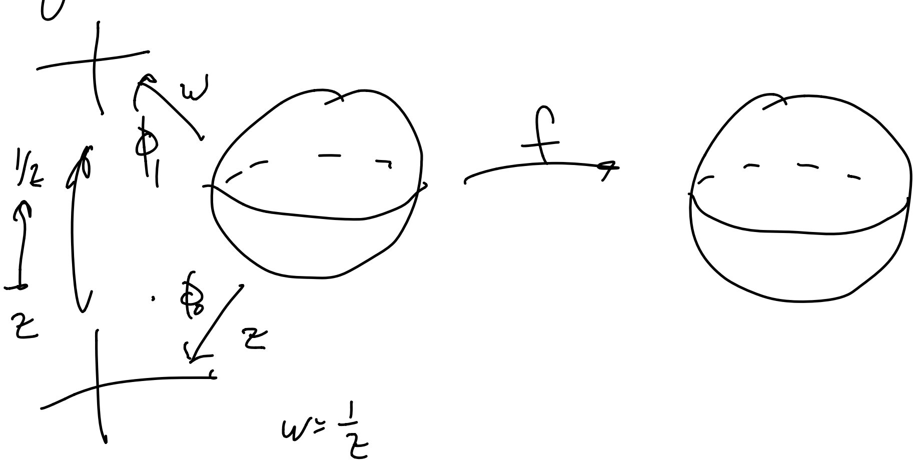
$$|f'(z)|_{Hyp} = \frac{f'(z) \cdot |\delta(f(z))|}{|\delta(z)|}$$

Automorphisms of the complex plane.

Theorem. A holomorphic function $f: \mathbb{C} \to \mathbb{C}$ is in $\mathrm{Aut}(\mathbb{C})$ if and only if $f(z) = az + b$ for some constants $a \neq 0$ and b

Proof. Say that $f: \mathbb{C} \to \mathbb{C}$ is holomorphic and injective. We can view f as a hol. map $\mathbb{C} \xrightarrow{f} \mathbb{C}_\infty$.

Is it possible to extend f a holomorphic function whose domain is $\mathbb{C}_\infty$?



If so then $f \circ \phi_1^{-1}$ would be meromorphic.
Write $F(w) = f(1/w)$.

$f \circ \phi_1^{-1}(w) = f(\frac{1}{w})$ would have a pole at $w=0$.

Now F is defined in $\{0 < |w| < 1\}$ and 0 is an *isolated* singular point.

It is either removable, a pole or an essential singularity. Which?
If it were essential then the image of the upper hemisphere $\{|z| > 1\} = \{|w| < 1\}$ would be dense in $\mathbb{C}$. But since f and hence F is injective the image of the upper hemisphere contains no points in the image of the lower hemisphere so 0 is not essential. In particular

∞ is a removable singularity

or a pole for F and

f does extend a holomorphic map

from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$ (taking ∞ to a finite pt. or ∞).

So $f(z) = \frac{P(z)}{Q(z)}$. When is a rational

map injective? $f(z) = w$ has a unique

solution?

$$\frac{P(z)}{Q(z)} = w \quad \text{so} \quad P(z) = wQ(z)$$

$$P(z) - wQ(z) = 0.$$

This has a unique solution if $\max\{\deg P, \deg Q\}$

$= 1$ so $\quad f(z) = \frac{az+b}{cz+d}$

Since f takes no finite z to ∞ $\quad f(z) = az + b$

as claimed

$$\frac{P'Q - PQ'}{Q^2}$$

Casorati-Weierstrass

$f(D-\{0\})$ avoids a nbd of w.

$z \mapsto \frac{1}{z-w}$ takes w to ∞.

$z \mapsto \frac{1}{f(z)-w}$ avoids a nbd. of ∞ so it is bounded.

Bdd. function has a removable singularity.

$$\frac{1}{f(z)-w} = g(z)$$

$$f(z)-w = \frac{1}{g(z)}$$

$f(z) = w + \frac{1}{g(z)}$ is mero-morphic or has a remov. sing.