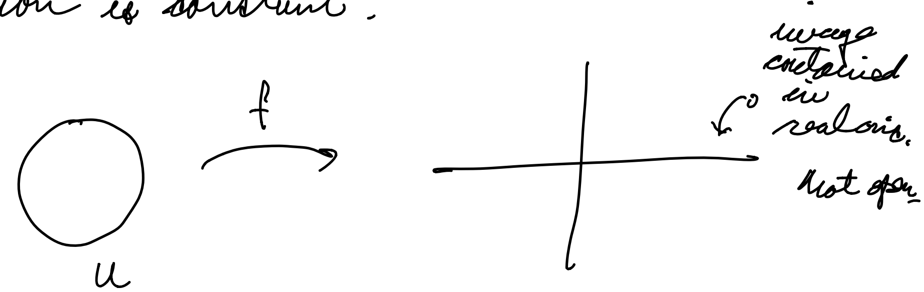
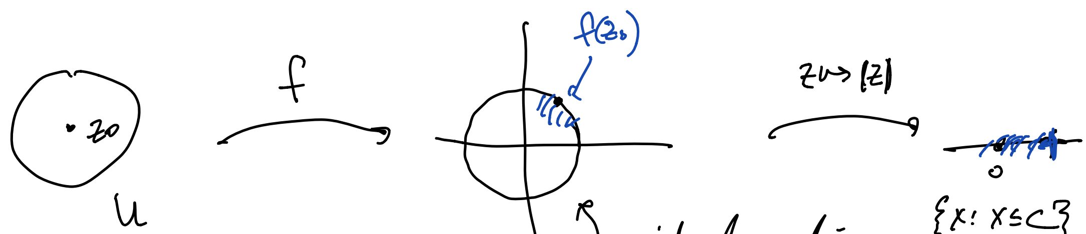
Open mapping theorem $\Rightarrow$ if the value of a holomorphic function is real then the function is constant.



If $|f(z)|$ has a local maximum at $z_0 \in U$ for some domain U then f is constant.



Let z_0 be a local max

Let $c = |f(z_0)|$

Recall that the valence $v(f, z_0)$ of a holomorphic function f at z_0 is n where

$$f(z) = f(z_0) + (z - z_0)^n g(z) \quad \text{with } g(z_0) \neq 0.$$

We can say $V(f, z_0) = \infty$ if f is locally constant at z_0.

Note that:

$$f'(z) = n(z - z_0)^{n-1} g(z) + (z - z_0)^n g'(z).$$

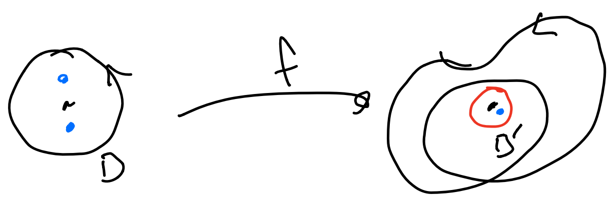
Follows that $f'(z_0) = 0$ iff $V(f, z_0) > 1$.

Also follows that $v(f', z_0) = v(f, z_0) - 1$.

Conclude that $v(f, z_0) =$ smallest $n \geq 1$ s.t. $f^{(n)}(z_0) = 0$.

Prop. If $f'(z_0) \neq 0$ then f is not locally injective.

Proof. Recall from the proof of the open mapping theorem that if $\nu(f, z_0) < \infty$ there is a disk D around z_0 and D' around $f(z_0)$



so that for $w \in D'$ we have

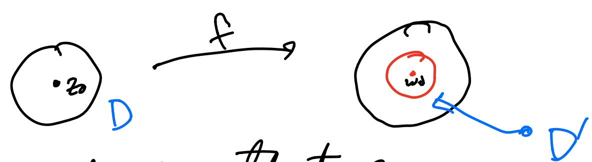
$$\sum_{z \in D : f(z) = w} \nu(f, z) = \nu(f, z_0) \quad (>1). \qquad *$$

Recall also that holomorphic functions have isolated zeros. Choose a smaller disk $D_0 \subset D$ in which $f'(z) \neq 0$ for $z \in D_0 - \{z_0\}$. Let D_0' be a smaller disk around $f(z_0)$ so that $*$ still holds for $w \in D_0'$. Now as long as $w \neq f(z_0)$ we have $\nu(f, z) = 1$ in $*$ so

$$\#\{z \in D' : f(z) = w\} = \nu(f, z_0) > 1. \quad \text{QED.}$$

Inverse function theorem. Say $f'(z_0) \neq 0$ then f has a local inverse.

Proof. Consider the disks as constructed



previously we see that as

$$f(z-z_0) = f(z_0) + (z-z_0)^n g(z) \qquad g(z_0) \neq 0$$

Now $f' \neq 0$ so $n = 1$.

We conclude that every point in D' has 1 preimage. Now consider the following modification of the winding number integrand

$$\frac{1}{2\pi i}\int_\gamma z\,\frac{f'(z)}{f(z)-w}\,dz \quad \text{vs.} \quad \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-w}\,dz.$$

$$\frac{1}{2\pi i}\int_{f(\gamma)} \frac{f^{-1}(z)}{} \quad ??$$

Where does this come from?

Pullback of

$$\frac{z\,dz}{z-w}$$

Cauchy's thm gives $\frac{1}{2\pi i}\oint_{f\gamma}\frac{z\,dz}{z-w} = w \cdot \text{wind}$

$$\int_\gamma \frac{f'(z)}{f(z)-w}\,dz = f^*\int_{f(\gamma)}\frac{dz}{z} = \text{wind}(f(\gamma), w)$$

Any f^{-1} exists as a function then

$$\int_{f(\gamma)} f^{-1}(z)\,\frac{dz}{(z-w)}$$

would give f^{-1} at $z=w$ or $f^{-1}(w)$ by Cauchy's thm.

$$f^*\left(\frac{f^{-1}(z)\,dz}{(z-w)}\right) = \frac{f^{-1}(f(z))\,f'(z)\,dz}{(f(z)-w)}$$

$$= \frac{z\,f'(z)\,dz}{f(z)-w}$$

$\int$

makes no explicit reference to $f^{-1}(z)$. Suggests (but does not prove) that this integrand is the one we want.

Prove it via residue calculation. (Exercise)

Claim that the left hand integral gives

$$\sum_{z_j : f(z)=w_0} z_j \cdot (\text{valence of } z).$$

mult. of z_j as a soln. of $f(z)=w_0$

Since there is only one solution and the multiplicity is 1 the integral is point z which maps to w. Holomorphicity of the inverse follows by differentiating (wrt w) under the integral sign.

$w \mapsto \int_\gamma f(z) \cdot \frac{f'(z)}{f(z)-w}\,dz$

Definition. A holomorphic map from $U^{\subset\mathbb{C}} \to V^{\subset\mathbb{C}}$ is <u>conformal</u> if $f'(z) \neq 0$ for $z \in U$.

Cor. A bijective conformal map has a conformal inverse.

Proof. Since f is bijective it has an inverse. The inverse map is locally holomorphic hence holomorphic.

Introduce Riemann surfaces

Definition. A *surface* S is a topological space together with a family

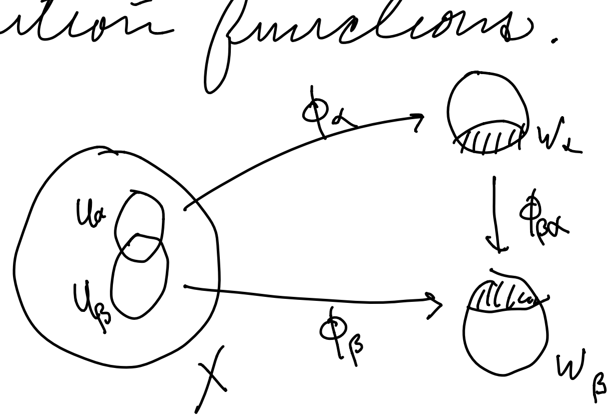
$\mathcal{A} = \{\phi_\alpha : \alpha \in \mathcal{A}'\}$ $\quad \phi_\alpha : U_\alpha^{\subset S} \longrightarrow W_\alpha^{\subset \mathbb{C}}$

so that

(1) Each ϕ_α is a homeomorphism of U_α onto an open set $W_\alpha \subset \mathbb{C}$.

(2) $\{U_\alpha : \alpha \in \mathcal{A}'\}$ is an open cover of S.

If U_α meets U_β then $\phi_{\beta\alpha} = \phi_\beta \circ \phi_\alpha^{-1}$ is a homeomorphism of $\phi_\alpha(U_\alpha \cap U_\beta)$ to $\phi_\beta(U_\alpha \cap U_\beta)$. The $\phi_{\beta\alpha}$ are called transition functions.



Thus:

$\phi_{\beta\alpha} \circ \phi_\alpha$

$= \phi_\beta \circ \phi_\alpha^{-1} \circ \phi_\alpha$

$= \phi_\beta$.

(where defined).

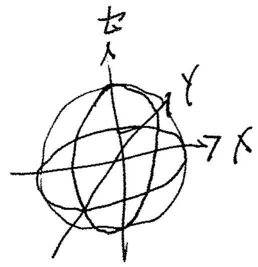
Two technical issues which I will momentarily defer.

Hausdorffness and whether we can take the index set to be countable.

Definition. A surface is a Riemann surface if the transition functions $\phi_{\alpha\beta}$ are holomorphic.

Riemann surface structure on S^2.

Let $S^2 = \{(x,y,t) : x^2+y^2+t^2 = 1\}$.



Let $NP = (0,0,1)$, $SP = (0,0,-1)$.

Use coordinates (x,y,t).

$U_1 = S^2 - NP$, $U_2 = S^2 - SP$.

$V_1 = \mathbb{C}$. Identify $\mathbb{C}$ with the x,y plane in $\mathbb{R}^3$.

coordinates by mapping $x+iy$ to $(x,y,0)$

For any pt. $P = (x,y,t) \neq NP$ draw the line from NP to P, intersect with the $t=0$ plane and identify $(x,y,0)$ with $x+iy \in \mathbb{C}$.

Formula for the line $s \mapsto (1-s)(0,0,1) + s(x,y,t)$.

Intersection parameter is s s.t. $(1-s)+st = 0$, $s = \frac{1}{1-t}$.

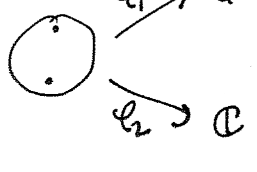
Intersection point is $\left(\frac{x}{1-t}, \frac{y}{1-t}, 0\right)$. Identify this with $\frac{x+iy}{1-t} \in \mathbb{C}$. So $\varphi_1(x,y,t) = \frac{x+iy}{1-t}$.

To define φ_2 we do the same construction starting with SP. In this case we use a different identification of $t=0$ with $\mathbb{C}$, send $(x,y,0)$ to $x-iy$ and get

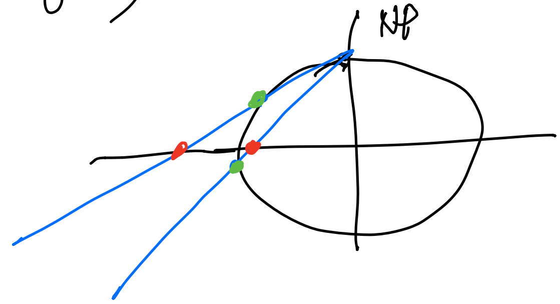
$\varphi_2(x,y,t) = \frac{x-iy}{1+t}$ defined on $U_2 = S^2 - SP$ $V_2 = \mathbb{C}$.

(If we had not done this we would have gotten an orientation reversing angle preserving map.)

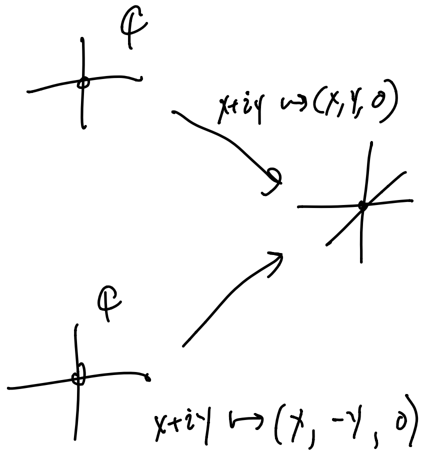
(Best to think of two distinct copies of $\mathbb{C}$)



(In the definition of the surface structure we can allow multiple copies of $\mathbb{C}$.)



$(x,y) \hookrightarrow (x,y,t)$



Neither of these maps is "holomorphic". The question of Holomorphicity first arises with respect to the overlap functions.

We calculate that for $p \overset{(x,y,t)}{=} \in S^2 - \{NP, SP\} = U_1 \cap U_2$

$\phi_1(p) \cdot \phi_2(p) = \frac{x+iy}{1-t} \cdot \frac{x-iy}{1+t} = \frac{x^2+y^2}{1-t^2} = 1.$ (complex mult.)

to product in $\mathbb{C}$.

since $x^2+y^2+t^2=1$, $x^2+y^2=1-t^2$.

complex #

Now $\phi_{21}(\phi_1(p)) = \phi_2(p) = \frac{1}{\phi_1(p)}$. inverse of complex #.

Solve for ϕ_{21} using the fact that it satisfies $\phi_{21} \circ \phi_1 = \phi_2$.

Write z for $\phi_1(p)$ then:

So setting $\phi_{21}(z) = \frac{1}{z}$ we have $\phi_{21} \circ \phi_1 = \phi_2$.

Since $z \mapsto \frac{1}{z}$ is conformal on $\mathbb{C} - \{0\}$ we have a conformal (or holomorphic) atlas.

Analogously $\phi_{12}(z) = \frac{1}{z}$.

$W_1 = \mathbb{C} - \{0\}$

$W_2 = \mathbb{C} - \{0\}$.