In Fridays discussion we focussed on the fact that the Riemann surface structure we defined extended to the vertices uniquely. We could ask the same uniqueness question about the edges. Example sheet problem.

(See page 22 of Forster.)

Proposition. Let R be a Riemann surface. Let $\tilde{R}$ be a covering space of R. Then $\tilde{R}$ has a natural Riemann surface structure for which the covering map is holomorphic.

Proof.

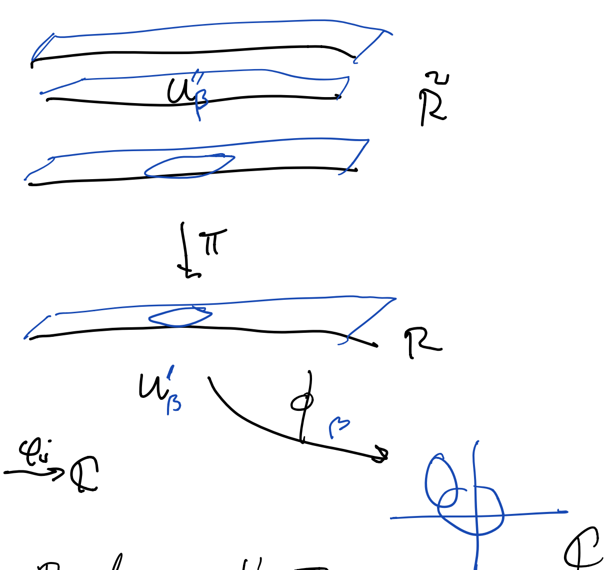
Start with an atlas $\mathcal{A}$ for R.

$\mathcal{A} = \{\phi_\alpha : U_\alpha \to W_\alpha \subset \mathbb{C}\}$. Now consider a modified atlas $\mathcal{A}'$. Consider open sets

U'_β with the property that U'_β is contained in some U_α and U'_β is evenly covered.

Let $\phi'_\beta = \phi_\alpha|U'_\beta$. Let $\mathcal{A}' = \{\phi'_\beta\}$.

Now define $\tilde{\mathcal{A}}$ on $\tilde{R}$.

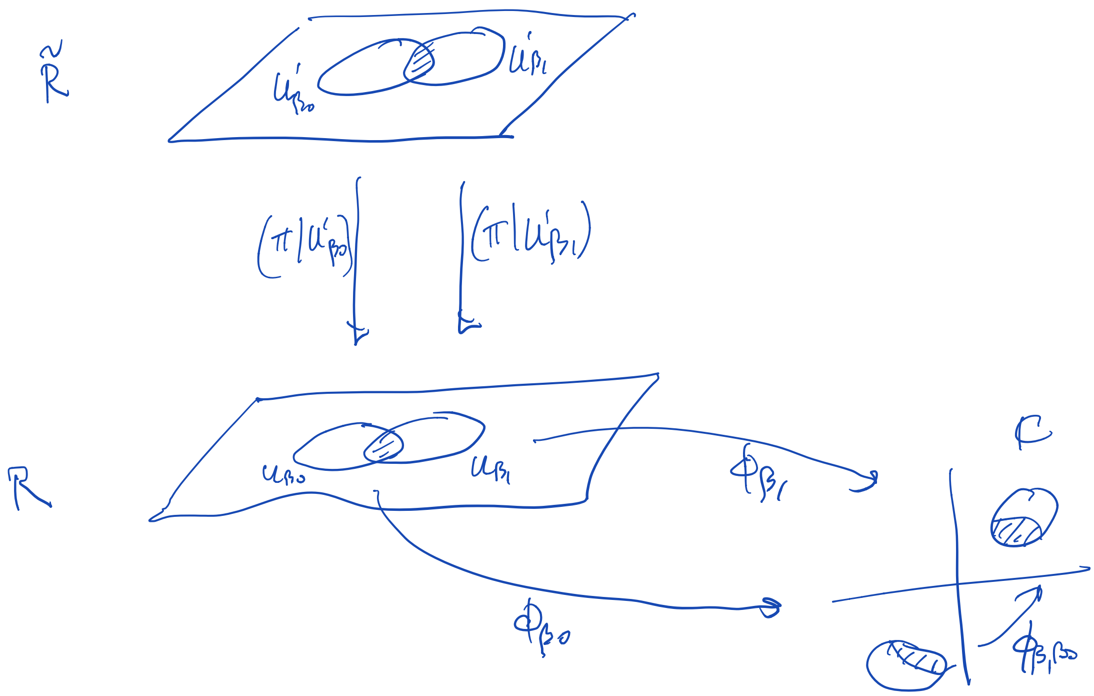


$U_\alpha \xrightarrow{\pi} U \xrightarrow{\phi_i} \mathbb{C}$

Let $\phi''_\beta : U''_\beta \to \mathbb{C}$ be $\phi'_\beta \circ \pi$.

$\tilde{\mathcal{A}} = \{\phi''_\beta\}$.

We now construct an atlas for $\tilde{R}$ where the open sets U'' are components of inverse images of sets U'_β. The charts are compositions $\phi''_\beta = \phi_\beta \circ \pi$ and the overlaps have the form $\phi_{\beta_1 \beta_0}$



$$\phi'_{\beta_1 \beta_0} = \phi'^{-1}_{\beta_1} \circ \phi_{\beta_0} = (\pi | U'_{\beta_1})^{-1} \circ \phi^{-1}_{\beta_1} \circ \phi_{\beta_1 \beta_0} \circ \phi_{\beta_0} \circ (\pi | U'_{\beta_0})$$

Recall that given a nice topological space X there is a correspondence between covering spaces $\tilde{X}$

of X and subgroups of $\pi_1(X)$. If we start with a Riemann surface R then we can use this abstract construction to build new Riemann surfaces.

(Weldon)

In the real case there is a topological issue with uniqueness of anti-derivatives

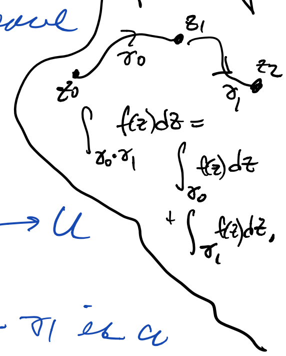
In the complex case there is a topological issue with existence of anti-derivatives.

Deal with this by passing to covering spaces.

Theorem. Path integration defines a homomorphism from $\pi_1(U, z_0)$ to $\mathbb{C}$ sending γ to $\int_\gamma f(z)\,dz$.

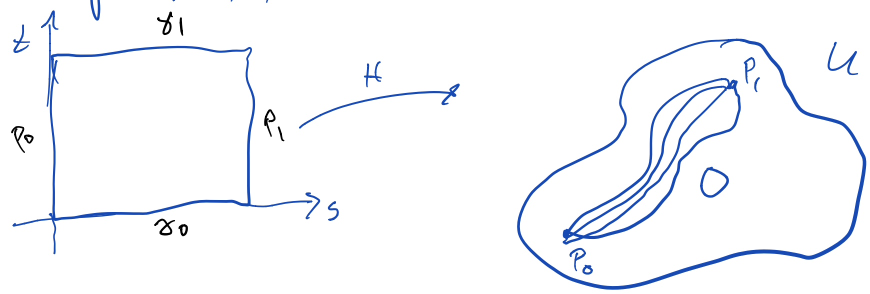
Additivity holds in a little more generality

Two parts to the proof. Homotopic paths have the same path integral. Additivity.



$$\int_{\gamma_0 \cdot \gamma_1} f(z)\,dz = \int_{\gamma_0} f(z)\,dz + \int_{\gamma_1} f(z)\,dz.$$

Homotopy invariance. Say that $\gamma_0 : [0,1] \longrightarrow U$, $\gamma_1 : [0,1] \longrightarrow U$. A homotopy from γ_0 to γ_1 is a map $H(s,t) : [0,1] \times [0,1] \longrightarrow U$



so that $H(s,0) = \gamma_0$

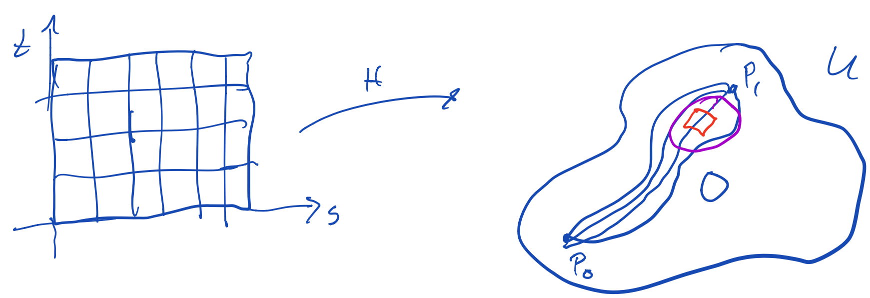
$H(s,1) = \gamma_1$

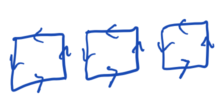
$H(0,t) = p_0$

$H(1,t) = p_1$.

We can cover U by disks on which f has an anti-derivative.

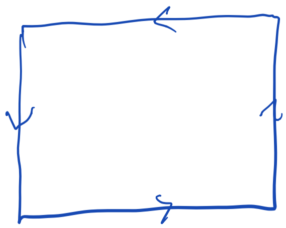
Divide the square into small squares so that the image of each small square is contained in such a disk.





Path integral around each small square is zero.

Path integrals along neighboring edges cancel.

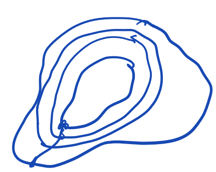


Result is that path integral around the boundary of the square vanishes.

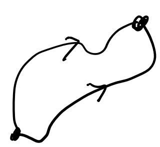
In the cohomology course you see 1-cochains which are functions from paths to coefficients. This is a good example.

(Also works if loops are freely homotopic.

Don't have to worry about basepoints.)



Recall the argument for constructing anti-derivatives locally. We use the fact that



integrals along two paths agree.

The condition that integrals along different paths agree is equivalent to the condition that integrals along loops are zero. Summary: anti-derivatives for f exist in a domain U if and only if $\oint_\gamma f\,dz = 0$ (for all loops). (iff $\int f : \pi_1(U) \to \mathbb{C}$ is 0)

This fits in neatly with the theory of covering spaces.

If this condition $\oint_{a} f dz$ does not hold on a domain U we can construct a unique minimal covering space where this condition does hold.