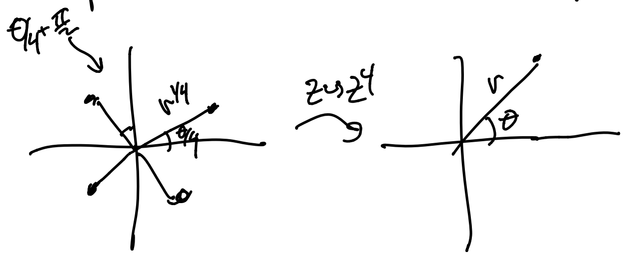
Let $f: R \to S$ be a holomorphic map between Riemann surfaces. For $q \in S$ we can count the number of points that map to it.

Let $n(q) = \# f^{-1}(q)$.

Example 1. $f(z) = z^n$. $\quad n(w) = \#\{z : z^n = w\} = n$ if $w \neq 0$

$\qquad\qquad\qquad\qquad 1$ if $w = 0$.



Let us define $d(q)$ to be the sum "with multiplicity".

$$d_f(q) = \sum_{p : f(p) = q} v_f(p).$$

In the previous example $v_f(p) = 1$ if $p \neq 0$ and n if $p = 0$ so

$$d_f(w) = n \text{ for } w \neq 0 \text{ as before}$$

but

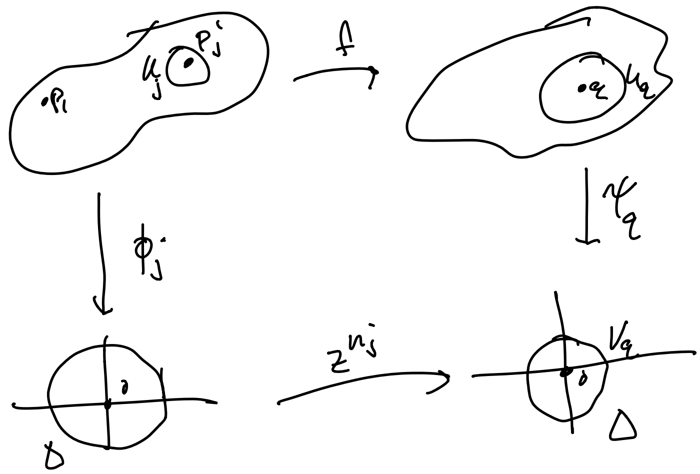
$$d_f(0) = v_f(0) = n \text{ as well.}$$

Example 2. Let $\iota : \Delta \to \mathbb{C}$ be the inclusion of the open unit disk. $d_\iota(w) = 1$ if $w \in \Delta$
0 $w \notin \Delta$.
Non constancy seems related to non-compactness of Δ?
(ι is an example of a <u>local homeomorphism</u> which is not a covering map.)

Theorem.
Let $f : R \longrightarrow S$ be a non-constant holomorphic map between compact connected Riemann surfaces. Then $d(q)$ is constant.

Proof. Remark inverse images of q are isolated so $\# f^{-1}(q)$ is finite by compactness.
It suffices to show that $d(q)$ is locally constant and then appeal to the connectivity of S.

Let $q \in S$. Choose a chart ψ_q around q which takes q to zero with image equal to the open unit disk.



Now around each p_j with $f(p_j)=q$ we have a nbd. U_j and a chart ϕ_j with $\phi_j^{-1}: U_j \to \mathbb{C}$ and $\phi_j(p_j)=0$ so that $\psi_q \circ f \circ \phi_j^{-1}(z) = z^{n_j}$ in Δ where $n_j = \nu_f(p_j)$ by our local model of holomorphic maps theorem.

We can assume the sets U_j are disjoint.

Let $E = R - \bigcup_j U_j$. E is closed hence compact.

Want to calculate $\sum_{f(p)=q'} V_f(p)$ for q' in a nbd. of q.

We get the contribution from each U_j and from E.

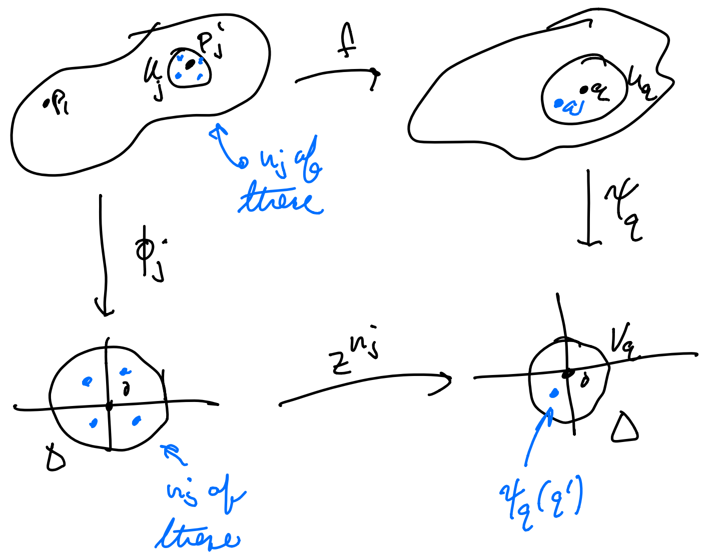
$$\sum_{f(p')=q'} V_f(p') = \left(\sum_j \sum_{\substack{p'\in D_j \\ f(p')=q'}} V_f(p') \right) + \sum_{\substack{p'\in E \\ f(p')=q'}} V_f(p')$$

Now E is closed so it is compact.
$f(E)$ does not contain q. Since it is compact there is a nbd. D'' of q disjoint from $f(E)$.
For $q'\in D''$ there is no contribution from E

so $$\sum_{f(p')=q'} = \sum_j \sum_{\substack{p'\in D_j \\ f(p')=q'}} V_f(p') \qquad = \sum_j V_f(p_j).$$

$$\sum_{\substack{p'\in D_j \\ f(p')=q'}} V_f(p') = \sum_{\substack{\phi_j(p')\in\Delta \\ \phi_j(p')^n = \psi_q(q')}} V_{f_{\alpha\beta}}(\phi_j(p')) = n = V_f(p_j).$$

Make use of the local picture.



So the sum depends only on q and not on q' near q. Thus $d_f(q')$ is locally constant hence globally constant.

Scholium. If $V_f(p) = 1$ for all p then f is a covering

In general f restricted to $R - f^{-1}(f(\{p\}))$ is a covering since the local model $z \mapsto z^n$ is a covering away from 0.

Definition. We call $d_f \equiv d_f(q)$ the *degree* of f. Note $d_f \geq 1$. (We saw before that a non constant hol. map is surjective, This is a refinement of that.).

Compare with smooth case: degree is defined in both cases. Smooth case we can count inverse images of a generic point by Sard's thm.

Here we can count inverse images of all pts.
Def. of degree requires a choice of orientation in our cases Riemann surfaces come with a choice of orientation. $d(f) \geq 1$ has no analogue in the general case.

Corollary. A meromorphic function on a compact. Riemann surface has the same number of zeros as poles (counted up to order).

Corollary. A holomorphic map between compact Riemann surfaces of degree 1 is a conformal equivalence.

Cor. If a merom. function on R has 1 zero of order 1 then R is conformally equivalent to S^2.

Proof. If $d_f = 1$ then every point has 1 inverse image. So f is invertible.

Inverse is holomorphic by the inverse function theorem.

Cor. If a compact Riemann surface R a meromorphic function with 1 simple zero then R is $\mathbb{C}_\infty$.

$R = \{(z,w) : w^2 = P(z)\}$ P has simple zeros.

$R \subset \mathbb{C} \times \mathbb{C} \hookrightarrow \mathbb{C}_\infty \times \mathbb{C}_\infty$.

Claim that $\overline{R} = R \cup \{(\infty,\infty)\}$. $\overline{R}$ is compact.
Is it a surface?

As $z \to \infty$, $|P(z)| \to \infty$

so $|w^2| \to \infty$ so $|w|^2 \to \infty$

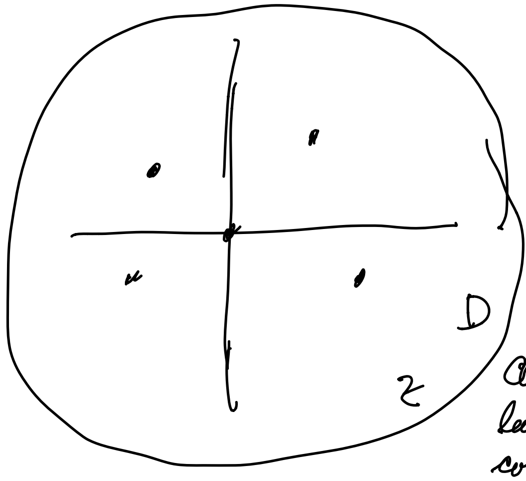
so $|w| \to \infty$.

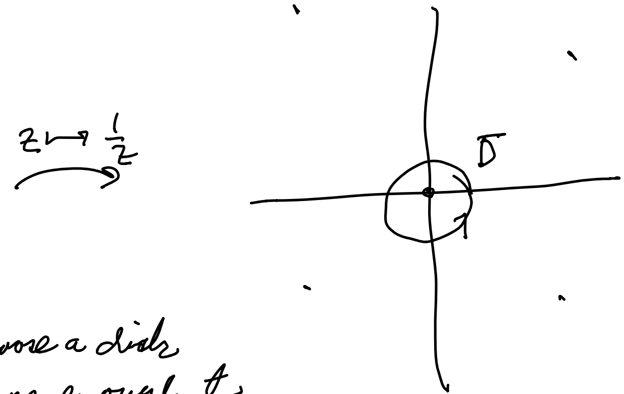
solution to the path lifting problem.

Defined a lifting function which
$\exp\left(\frac{1}{2}\int_\gamma \frac{P'(z)}{P(z)}\, d\gamma\right)$

gave us a parametrization over any simply connected set.

Can we use this to get a parametrization over a nbd of ∞? A nbd. of ∞ is <u>not</u> simply connected. Punctured disk.





Choose a disk large enough to contain all the zeros of P

Switch to $\frac{1}{z}$ coordinates and view the lifting maps as going from $\frac{1}{z}$ to $\frac{1}{w}$ coordinates.

Consider the punctured disk $\bar{D}$.

In order to use path lifting to define a coordinate unambiguously we want $\exp \frac{1}{2}\int_\gamma \frac{P'}{P}dz$ to be equal to 1 for every loop.

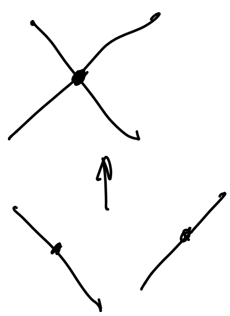
This will be the case if it holds for the generator of the fundamental group.

This will be true if deg P is even.

But note that the value of the function depends on our choice of an initial solution.

We get 2 hol. maps from the punctured disk to the punctured disk. Each has a cont. extension to the disk so each has a holomorphic extension.

Local picture near (∞,∞) in $\mathbb{C}_\infty \times \mathbb{C}_\infty$ is



We define a Riemann surface structure so that these disks are disjoint $\tilde{R}$. We get a hol. map $\phi : \tilde{R} \to R$ which takes these 2 pts. to 1 pt.

If deg P is odd then