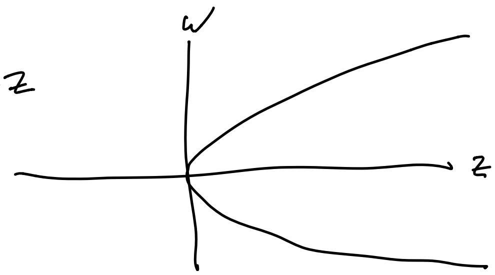
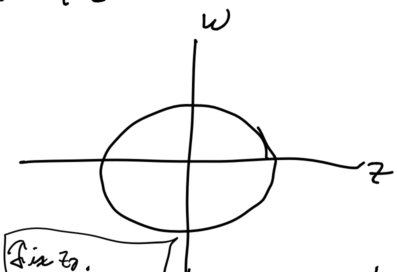
If P has real coefficients we can draw the corresponding real locus:

$w^2 = z$



$w^2 = 1 - z^2$



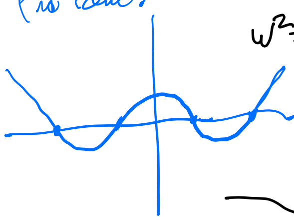
Fix z_0.

Say z_0 is real and P is real. If $P(z_0) > 0$ then there are two real points $(z_0, \sqrt{P(z_0)})$ $(z_0, -\sqrt{P(z_0)})$.

If $P(z_0) = 0$ there is 1 real solution $(z_0, 0)$.

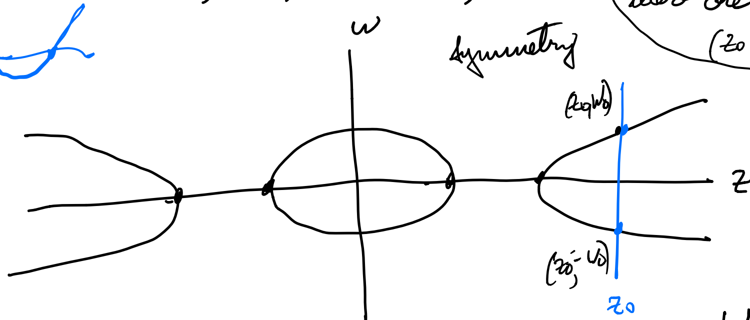
If $P(z_0) < 0$ no real solutions.

Real picture when P is real.

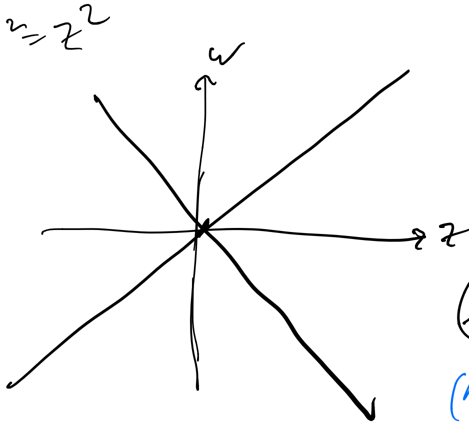


$w^2 = (z+2)(z+1)(z-1)(z-2)$

symmetry

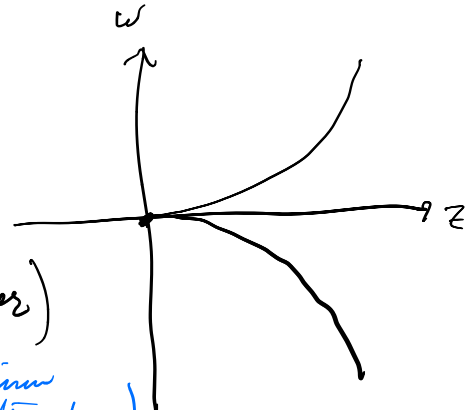


$w^2 = z^2$



(singular)

$w^2 = z^3$



(Not Riemann surfaces though...)

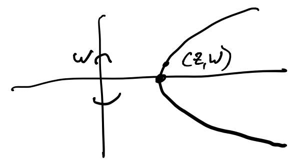
To give these varieties Riemann surface structures we construct charts (in the real or complex setting) by projecting onto the z or w axes and using the implicit function theorem to write

our variety locally as the graph of a function, writing either z as a function of w or w as a function of z.

We will not describe the general case we only look at the hyper-elliptic case.

Prop. If P has simple zeros then R has a Riemann surface structure.

Charts have the form $\pi_z((z,w)) = z$ or $\pi_w((z,w)) = w$. We show that these charts are homeomorphisms by constructing inverse charts ie locally solving for w as a function of z or locally solving for z as a function of w.



z is locally a function of w

$z = P^{-1}(w^2)$ ~~local~~

inverse in the sense of ~~inverse~~ function ~~locally~~ the graph of a function. $(z, \sqrt{P(z)})$

Any $(z_0, w_0) \in V$.

When $P(z_0) \neq 0$ we use the chart π_z in a nbd. of (z_0, w_0). We get an inverse chart by solving for w as a function of z.

$w^2 = P(z)$, $w = \sqrt{P(z)}$.

Specifically, choose some disk D centered at $P(z_0)$ and not containing 0. Let $\psi: D \to \mathbb{C}$ be a branch of $\sqrt{z}$ so $\psi^2(z) = z$.

Consider $z \mapsto (z, \underbrace{\psi \circ P(z)}_{w})$.

satisfies $w^2 = (\psi \circ P(z))^2 = \psi^2(P(z)) = P(z)$ so in V.

Thus our chart is a homeomorphism (in fact a holomorphic map from U to $\mathbb{C}^2$.)

When $P(z_0) = 0$ we use the chart $\pi_w(z,w) = w$.

To show that this chart is a homeomorphism in a nbd of (z_0, w_0) we need to solve for z as a function of w. $w^2 = P(z)$ $P^{-1}(w^2) = P^{-1}(P(z)) = z$.

We use 2 facts. First since $P(z_0) = 0$ z_0 is a simple root of P,

$$P = c \cdot (z - z_0) \cdot (z - z_1)(z - z_2)$$

$$P'(z) = c(z - z_1) \cdots$$

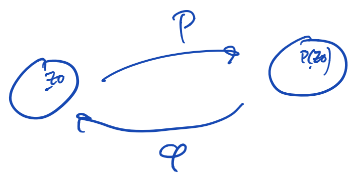
$$+ c(z - z_0) \cdot (z - z_2) \cdots$$

$$+ \vdots$$

so $P'(z_0) \neq 0$.

Secondly we use the inverse function theorem: if $P'(z_0) \neq 0$ then P' has a local inverse φ defined in a nbd. of $P(z_0)$. $P \circ \varphi(z) = z$

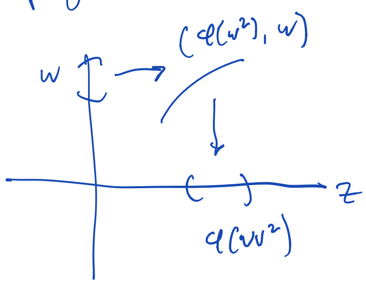
for



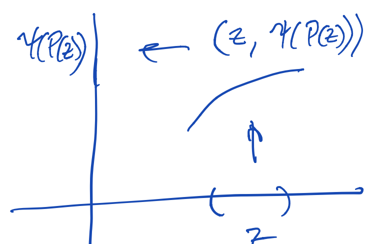
Consider $w \longmapsto (\underbrace{\varphi(w^2)}_{= z}, w)$. $P(z) = P \circ \varphi(w^2) = w^2$.

This is a local inverse to π_w.

Overlap functions



and



are holomorphic.

Note that we have in fact shown that away from the zeros of P the map $\pi_z : R \to \mathbb{C}$ is a covering map of degree 2.

$U = \mathbb{C} - \{z : P(z) = 0\}$.

$R' = \{ w^2 = P(z) : P(z) \neq 0 \}$

This is true without any hypothesis.

This is a regular cover. The deck group is $\mathbb{Z}/2\mathbb{Z}$ and is generated by the involution that takes (z,w) to $(z,-w)$.

Our next objective is to get a "global" picture of these surfaces as topological objects. In the real case this would involve counting components and determining which are circles and which are open intervals. For surfaces we want to look at the genus and the number of "punctures".

A compact orientable surface is determined by its genus:

$g=1$ $\quad$ $g=2$ $\quad$ $g=3$ $\quad$ ...

We will see that our surfaces V can be described as compact orientable surfaces with a finite number of points removed.

In order to determine the type of our surfaces we want to find a more explicit construction of construction of the "function" $w = \sqrt{P(z)}$ which allowed us to solve for w in terms of z.

Motivation: $\sqrt{P(z)} = \exp\left(\frac{1}{2} \log P(z)\right)$

$$= \exp\left(\frac{1}{2} \int_\gamma \frac{d}{dz} \log P(z)\right)$$

$$= \exp\left(\frac{1}{2} \int_\gamma \frac{P'(z)}{P(z)}\, dz\right).$$

Now we should justify our intuition.

Let $U^* = \mathbb{C} - \{\text{roots of } P\}$, $\quad V^* = \{(z,w) : w^2 = P(z),\ w \neq 0\}$

$\pi_z : V^* \to U^*$ is a covering map.