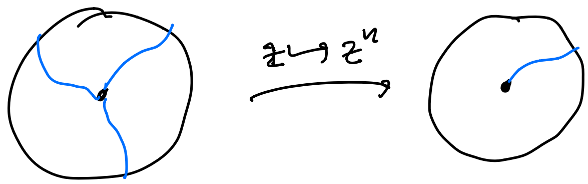
Comment. In the following proof we use the existence of triangulations of surfaces. These *do* exist but we are not proving that they exist.

Riemann–Hurwitz Theorem. Let $f: R \to S$ be a non-constant holomorphic map between compact Riemann surfaces. Then

$$(1) \quad \chi(R) = d \cdot \chi(S) - \underbrace{\sum_{\substack{p \in R \\ V_f(p) > 1}} \left(V_f(p) - 1\right)}_{\text{sign is determined}}.$$

Proof. There are only finitely many points $p \in R$ with $V_f(p) > 1$.

Construct a triangulation of S where the set of vertices $q_1 \dots q_k$ contains the images under f of $p_1 \dots p_j$. Away from the inverse images of the q's the map is a covering map of degree d so each open edge lifts to d open edges in R.

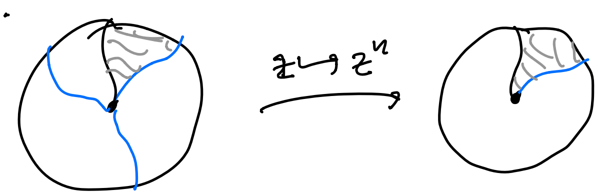


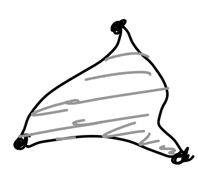
Can use the local model of the map to show that we can extend the lift of the path to its endpoints:

$$\begin{tikzcd} & R - \{p_1 \dots p_j\} \arrow[d, "f"] \\ \sigma : [0,1] \arrow[ur, "\tilde{\sigma}"] \arrow[r] & S - \{q_1 \dots q_k\} \end{tikzcd}$$

If $\lim_{t \to 0} \sigma(t) = 0$ then $\lim_{t \to 0} \sigma^{1/n}(t) = 0$.

Each triangle downstairs lifts to d topological triangles upstairs since triangles are simply connected.





The Euler characteristic can be calculated as the alternating sum of #'s of simplices in a triangulation.

$$\chi(S) = \# v(S) - \# e(S) + \# f(S)$$

$$\chi(R) = \# v(R) - \# e(R) + \# f(R)$$

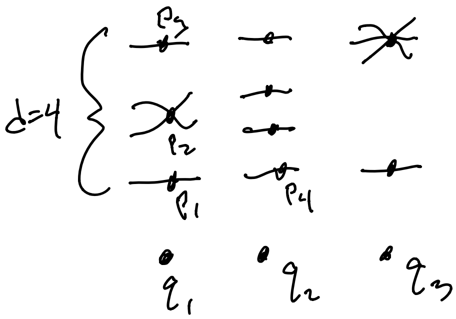
Now $\# e(R) = d\, \# e(S)$ and $\# f(S) = d \cdot \# f(R)$

$$\chi(R) - d\,\chi(S) = \# v(R) - d \cdot \# v(S).$$

$$= \sum_{q \in v(S)} \left(\# f^{-1}(q) - d \right)$$

Now recall that $\displaystyle\sum_{p : f(p) = q} v_f(p) = d$

So $$\chi(R) - d\,\chi(S) = \sum_{q \in v(S)} \left(\# f^{-1}(q) - \sum_{p : f(p) = q} v_f(p) \right)$$



$$= \sum_{q\in V(S)} \left(\sum_{p: f(p)=q} 1 - \sum_{p: f(p)=q} v_f(p) \right)$$

$$= \sum_{q\in V(S)} \sum_{p: f(p)=q} \left(1 - v_f(p)\right)$$

$$= \sum_{p\in V(R)} \left(1 - v_f(p)\right)$$

$$= \sum_{p\in R} 1 - v_f(p).$$

Note that points p with $v_f(p)=1$ make no contribution.

Makes no explicit connection with the triangulation.

Cor. If $f: R \to S$ is hol. and non constant then $g(S) \le g(R)$.

Recall $\chi(R) = 2-2g$. Typically $\chi < 0$.

$\chi=2$, $\chi=0$, $\chi=-2$ ... $\chi=-4$ ...

$g=0$ $g=1$ $g=2$ $g=3$

Proof. It is sometimes useful to rewrite the Riemann-Hurwitz equation in terms of the genus.

$1-g = \frac{\chi}{2}$ $\quad g = \frac{\chi}{2}+1$

$-g = \frac{\chi}{2}-1$

$$2-2g(R) = d(2-2g(S)) - \sum V_f(p)-1.$$

$$2g(R)-2 = \underset{\geq 1}{d}(2g(S)-2) + \underbrace{\sum V_f(p)-1}_{\geq 0}$$

If $S=S^2$ then the assertion is true.

If $S \neq S^2$ then $2g(S)-2 \geq 0$ so

$$2g(R)-2 \geq d(2g(S)-2) \geq 2g(S)-2$$

$$g(R) \geq g(S)$$

Compactification discussion.

We introduced hyper-elliptic surfaces as non-compact surfaces in $R \subset \mathbb{C}^2$.
We have seen how useful it is to deal with compact surfaces.
(mero. fcns. on $\mathbb{C}_\infty$, Riemann-Hurwitz)

I will now show that for each R there is a compact Riemann surface $\bar{R}$ so that R is conformally equivalent to $\bar{R}$ - one or two points.

We start by "completing" $\mathbb{C}^2$.
We "completed" $\mathbb{C}$ to $\mathbb{C}_\infty$.

Consider $\quad \mathbb{C}^2 = \mathbb{C} \times \mathbb{C} \subset \mathbb{C}_\infty \times \mathbb{C}_\infty$.

Let R^+ be the closure of R in $\mathbb{C}_\infty \times \mathbb{C}_\infty$.

$R = \{(z,w) : w^2 = P(z)\}$. Recall that as

$z \to \infty, \quad P(z) \to \infty, \quad |P(z)| \to \infty, \quad |w| \to \infty, \quad w \to \infty$.

It follows that R^+ consists of $R \cup \{(\infty,\infty)\}$.

Now we choose local coordinates near (∞,∞).

$$
\begin{aligned}
\text{Let} \quad \psi(z_1, w_1) &= \left(\tfrac{1}{z}, \tfrac{1}{w}\right) && \text{if } z, w \neq \infty \\
&= \left(\infty, \tfrac{1}{w}\right) && \text{if } z = 0 \\
&= \left(\tfrac{1}{z}, \infty\right) && \text{if } w = 0 \\
&= (\infty, \infty) && \text{if } z = w = 0.
\end{aligned}
$$

Since we are studying the local structure of R^+ near (∞,∞) it is equivalent to study the pullback $\varphi^{-1}(R^+)$ in terms of the coordinates z_1, w_1.

This set is determined by the equation

$$w^2 = P(z)$$

$$\left(\frac{1}{w_1}\right)^2 = P\left(\frac{1}{z_1}\right)$$

or $$w_1^2 = \frac{1}{P\left(\frac{1}{z_1}\right)}$$

Now

$P(z) = a_0 + a_1 z + \dots a_d z^d$ with $a_d \neq 0$

$$P\left(\frac{1}{z}\right) = a_0 + \frac{a_1}{z} + \dots \frac{a_d}{z^d}$$

$$= \frac{1}{z^d}\left(a_d + a_{d-1} z + \dots a_0 z^d\right)$$

$$\frac{1}{P(1/z)} = \frac{z^d}{a_d + a_{d-1} z + \dots a_0 z^d} = z^d \cdot g(z) \text{ with } g(0) \neq 0$$

It follows that $z \mapsto \frac{1}{P(1/z)}$ extends to

$z \mapsto h(z)$

a holomorphic function taking 0 to 0 and the valence $V_h(0) = d$.

In these new coords R^+ becomes

$$R^+ = \{(z_1, w_1) : w_1^2 = h(z_1)\} \quad \text{near } (\infty,\infty).$$

Now we could work in the new coords (z_1, w_1) and not keep track of the explicit change of variables.

Coord. change is holomorphic isomorphism.

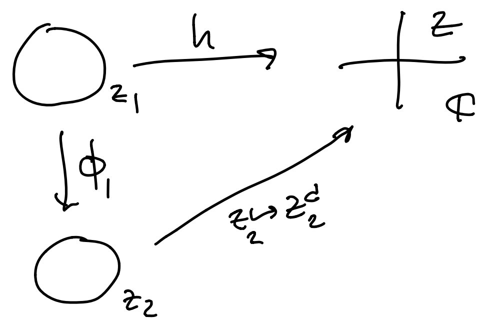
In fact we will change variables one more time to make the equation simpler. Recall that locally a a holomorphic function h with $h(0)=0$ looks like $z \mapsto z^n$ n = valence of h at 0.

Now we can introduce a new variable

$z_2 = \phi_1^{-1}(z_1)$

$G(z_2) = z_2^d$.

$G(\phi_1^{-1}(z_1)) = \phi_1^{-1}(z_1)^d$.



$z_1 = \phi_1^{-1}(z_2)$

$\phi_1(z_1) = z_2$

$z = \phi_1^d(z_1)$

In these coords. R looks like $\quad w_1^2 = z_2^d$.

R^+ looks like $\{w_1^2 = z_2^d\}$.

$z = z_2$.

$h(z_1)$

$= \phi_1^d(z_1)$

$h(\phi_1^{-1}(z_2)) = z_2^d$