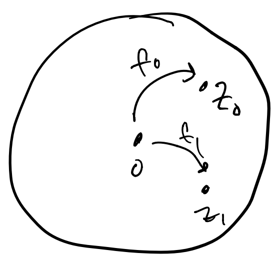
PGL(2, $\mathbb{C}$) group of Möbius transformations

$\mathcal{G} = \left\{ \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix} : |a|^2 - |c|^2 = 1 \right\}$.

$\mathcal{G}$ acts transitively on Δ:

(Given



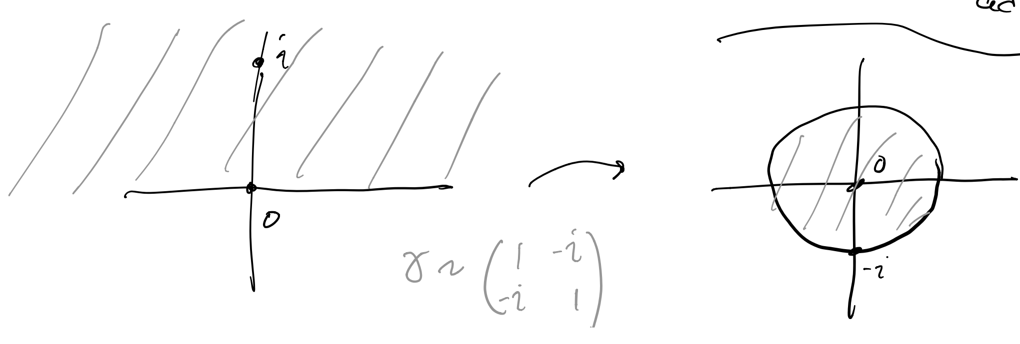
$f \in \mathcal{G}$ $f(0) = z$.

Given any 2 points z_0, z_1 there is some $f \in \mathcal{G}$ taking z_0 to z_1, $f_1 \circ f_0^{-1}$.)

There is an element of PSL(2, $\mathbb{C}$) that takes Δ to the upper half plane: UHP.

Conjugating by this element takes $\mathcal{G}$ to PSL(2, $\mathbb{R}$).

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$.



$0 \mapsto -i$

$\infty \mapsto +i$

$i \mapsto 0$

$z \mapsto \dfrac{az+b}{cz+d}$

$\dfrac{b}{d} = -i$

$\dfrac{a}{c} = i$

$ai + b = 0$

$z \mapsto \dfrac{z-i}{-iz+1}$

$\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$

$b = -i$

$a = 1$

$\dfrac{1}{c} = i \quad c = -i$

$d = 1$

$\dfrac{z-i}{-iz+1}$

$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = ($$

$$\begin{pmatrix} a+ci & \bar{c}+i\bar{a} \\ ai+c & i\bar{c}+\bar{a} \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

$$(a+ci - i\bar{c} + \bar{a}$$

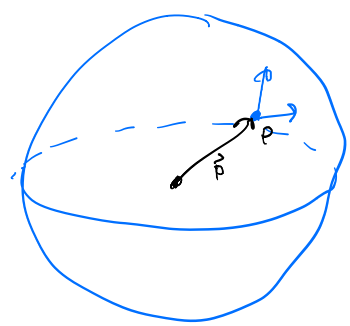
In the course of our argument characterizing $\mathrm{Aut}(\Delta)$ we showed:

$f \sim \begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix}$

$$(*) \quad 1 - |f(z)|^2 = (1 - |z|^2) \cdot \frac{1}{|cz+\bar{a}|^2}$$

How do we describe geometry on a 2-manifold?

(P, V)



$T(S^2) = \{(P,V) : \vec{P}\cdot\vec{V} = 0\}$.

$|\vec{P}|^2 = 1$

Geometry of a surface is described once you know the lengths of tangent vectors. If you know this you also know the lengths of paths

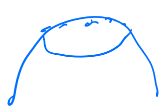
$\gamma : [a,b] \to S^2 \qquad \text{length}(\gamma) = \int_a^b \text{length}\left(\frac{d\gamma}{dt}\right) dt.$

You know the distance between points

$d(p,q) = \inf(\text{length } \gamma : \gamma(0)=p,\ \gamma(1)=q)$.

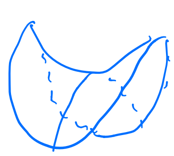
Most important invariant of a surface is the <u>curvature</u>.

Example: The sphere has positive curvature. Has the property that



the circumference of a circle of radius r increases more slowly than the corresponding Euclidean circle. ($< 2\pi r$)

Example: Hyperboloid of 1-sheet has negative curvature.



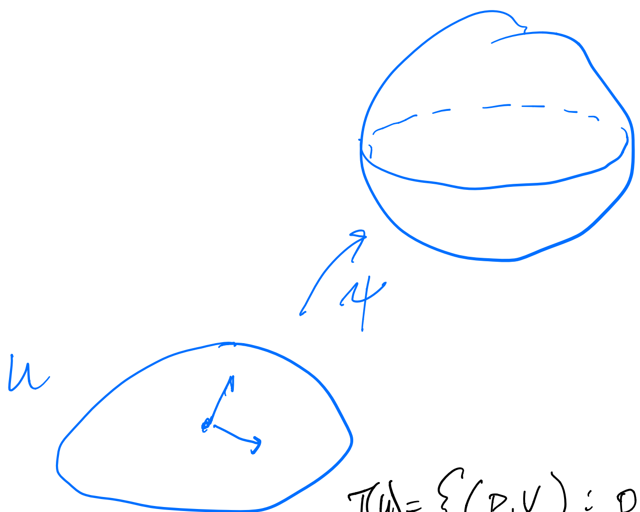
"saddle"

Circumference of a circle of radius r increases more quickly than a Euclidean circle.

How do we describe geometry in a surface?

Pull back the metric on the surface in $\mathbb{R}^3$ to the chart.

Delete this?



$T(U) = \{(p, v) : p \in U,\ v \in \mathbb{R}^2\}.$

Can express the pulled back form in terms of coord. fns $dx(v)$, $dy(v)$.

$$I(v,v) = E\,(dx(v))^2 + 2F\,(dx(v)\cdot dy(v)) + G\,(dy(v))^2$$

$$I = E\,dx^2 + F\,dx\cdot dy + G\,dy^2.$$

Quadratic form determines the bilinear form.

$E \rightsquigarrow (\text{length of } \partial_x)^2$

$G \rightsquigarrow (\text{length of } \partial_y)^2$

$F \rightsquigarrow$ angle between ∂_x and ∂_y.

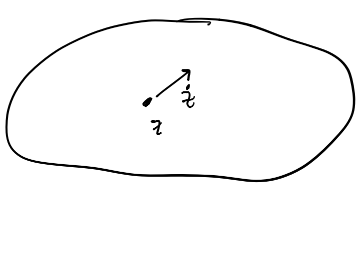
Also written in terms of arclength ds

$$(ds)^2 = E\,dx^2 + 2F\,dx \cdot dy + G\,dy^2.$$

or $ds = \sqrt{E\,dx^2 + 2F\,dx \cdot dy + G\,dy^2}$.

Now say that our surface is a <u>Riemann surface</u>.

$T(U) = \{(z, \dot{z}) : z \in U \ \dot{z} \in \mathbb{C}\}$.



As a Riemann surface we already have a notion of the angle between two vectors we can ask whether the new metric we are

putting on $U \subset \mathbb{C}$ gives the same notion of angle. We say coordinate charts are <u>isothermal</u> if this is the case. (Gauss proved the existence of isothermal charts.

Example. Polygons
$E, G = 1$ $F = 0$.

To say that two metrics give the same notion of angle means they one is a scalar multiple of the other. To say that the first fundamental form is a multiple of the Euclidean metric means $E = G$ and $F = 0$.

Def. We say that a metric on a Riemann surface is conformal if it can be written in charts as $ds^2 = \varphi(z) \cdot |dz|^2 = \varphi(z)(dx^2 + dy^2)$

ie $E = G = \varphi$, $F = 0$.

Compare with flat metric on $\mathbb{C}$.

$E = G = 1 \quad F = 0$. flat metric on U.

So charts are iso-thermal if the pulled back metric (expressed by the first fundamental form) is conformal.

Example. We constructed charts for S^2 ϕ_1, ϕ_2. Let ψ_1, ψ_2 be the corresponding inverse charts. If we calculate the first fundamental form for ψ_1 we get (mathematica)

$$E = G = \frac{4}{(1+|z|^2)^2} \qquad F = 0$$

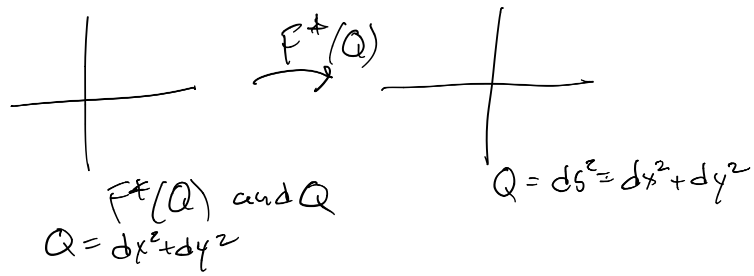
$$ds^2 = \frac{4}{(1+|z|^2)^2} |dz|^2$$

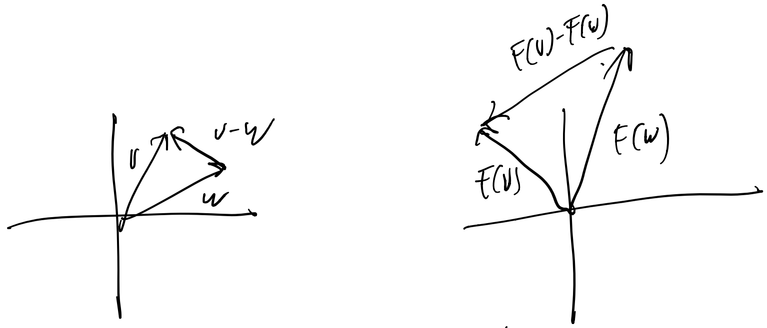
$$ds = \frac{2}{(1+|z|^2)} |dz|.$$

Rescaling the metric by $\varphi = \frac{2}{(1+|z|^2)}$.

Proposition. Let Q be a pos. def. bilinear form. $F^*(Q)$ and Q forms give the same notion of angle if and only if one is a scalar multiple of the other.

Proof.





If F preserves angles then these triangles are similar so $|F(v)| = \lambda\cdot|v|$, $|F(w)| = \lambda\cdot|w|$.

$F^*(Q) = \lambda\cdot Q$.

Can read this differently.

(Pulling back metrics versus pulling back complex structures.)

The geometry of $\mathbb{R}^3$ determines a metric on S^3 or P.

Metric plus orientation determines a complex structure on the tangent spaces.

Such a structure is called an "almost complex structure". We have built a Riemann surface structure which is compatible with this almost complex structure.

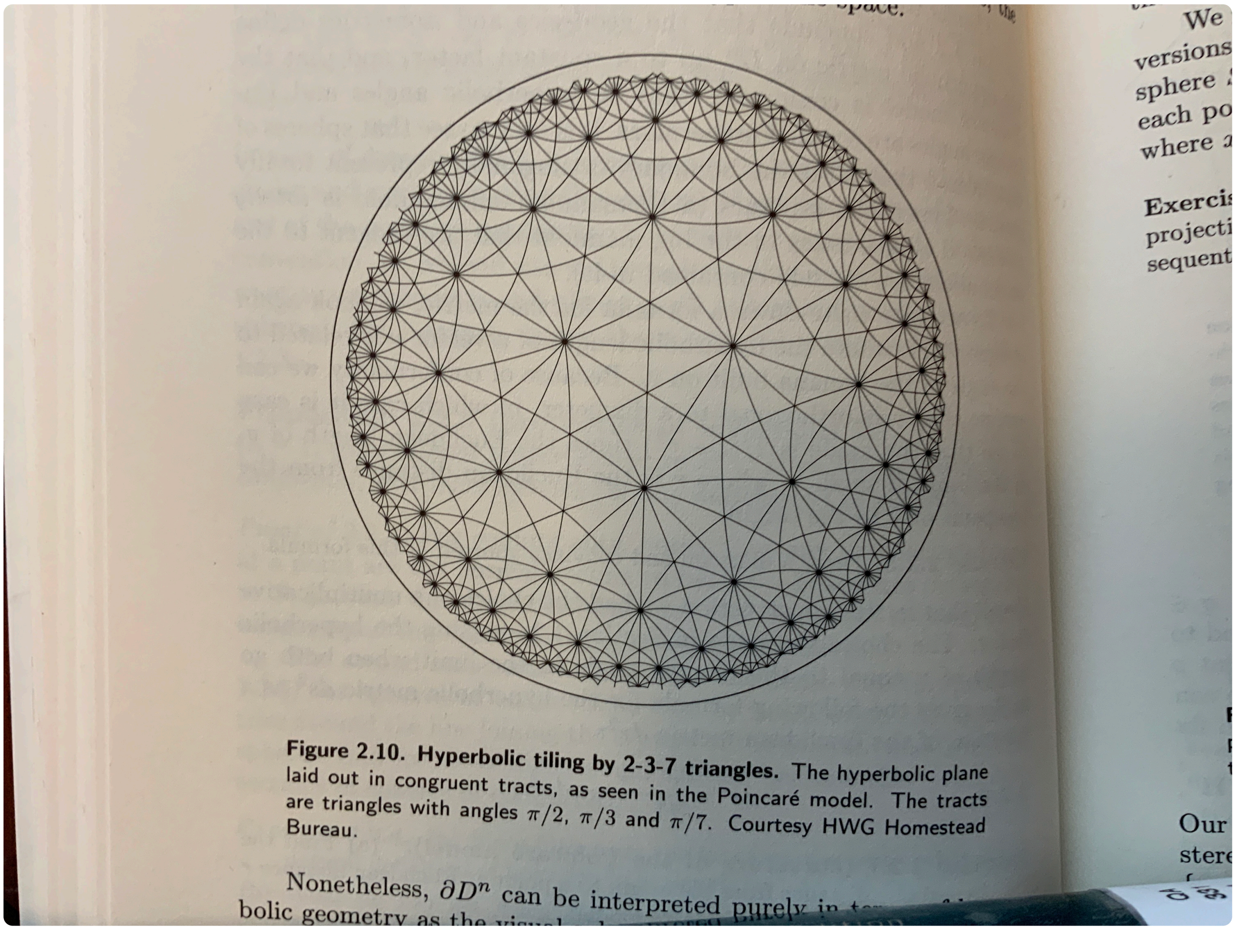


Figure 2.10. Hyperbolic tiling by 2-3-7 triangles. The hyperbolic plane laid out in congruent tracts, as seen in the Poincaré model. The tracts are triangles with angles $\pi/2$, $\pi/3$ and $\pi/7$. Courtesy HWG Homestead Bureau.

Nonetheless, ∂D^n can be interpreted purely in
bolic geometry as the

Picture credit: W. Thurston Three-dimensional geometry and topology.

Remark. Circumference of a circle grows <u>exponentially</u> with the radius.

to its Euclidean distance from the bounding p
shows the same congrent tracts as figure 2.10, but seen in the upper half-space model.

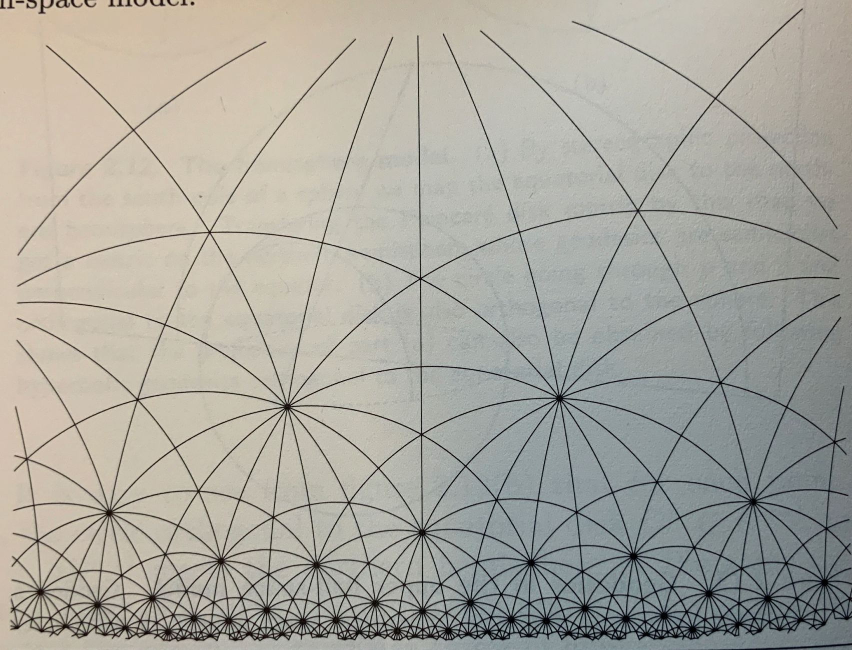
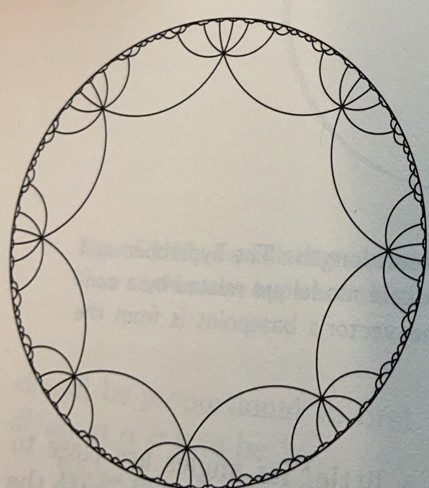


Figure 2.14. Hyperbolic tiling by 2-3-7 triangles. Another view of the hyperbolic world divided into congruent tracts. Upper half-plane projection.

are in the model—in particular, they quickly start looking very small as we move away from the origin—but they can all be obtained from one another by hyperbolic isometries. For example, the two copies in Figure 1.13(b) are mapped to one another by a reflection in L, followed by a reflection in M.



(a)