We start by recalling the compactification discussion from last time.

$$
\begin{array}{cccccc}
 & (0,0) & \longmapsto & (0,0) & \longmapsto & (\infty,\infty) \\
\mathbb{C}\times\mathbb{C} \supset & D_r \times D_r & \xrightarrow{\Psi_1} & \mathbb{C}\times\mathbb{C} & \xrightarrow{\Psi_0} & \mathbb{C}_\infty \times \mathbb{C}_\infty \\
 & (z_2, w_1) & & (z_1, w_1) & & (z, w) \\
 & \{w_1^2 = z_2^d\} & & \{w_1^2 = h(z)\} & & \{w^2 = P(z)\} \\
 & (\Psi_0 \circ \Psi_1)^{-1}(R) & & \Psi_0^*(R) & & R
\end{array}
$$

Using the fact that $v_h(0) = d$

Now we can introduce a new variable

$h(z_1) = \phi_1(z_1)^d \quad (*)$

$$
\begin{tikzcd}
\bigcirc_{z_1} \arrow[r, "h"] \arrow[d, "\phi_1"] & \mathbb{C}_z \\
\bigcirc_{z_2} \arrow[ur, "z_2 \mapsto z_2^d"'] &
\end{tikzcd}
$$

$z_1 = \phi_1^{-1}(z_2)$

$z = \phi_1^d(z_1)$

Introduce a new variable z_2 where

$$z_2 = \phi_1(z_1)$$

and

$$z_1 = \phi_1^{-1}(z_2).$$

$(*)$ becomes $\quad h(z_1) = z_2^d$.

The equation $\quad w_1^2 = h(z_1) \quad$ is expressed in these new coordinates as $\quad w_1^2 = z_2^d$.

Analyze this equation:

Case I.

Assume d is even. Then $\quad w_1^2 = z_2^{2k}$

$d = 2k$

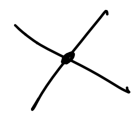
$$w_1^2 - z_2^{2k} = 0$$

$$(w_1 - z_2^k)(w_1 + z_2^k) = 0$$

so $\quad (w_1 - z_2^k) = 0$ or $(w_1 + z_2^k) = 0$.

$\qquad$ branch 1 $\qquad\qquad$ branch 2

Our variety has 2 branches meeting at the point $(0,0)$.



"node"

We can define an inverse chart for each branch

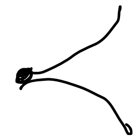
$$\psi_1 : D_r \longrightarrow \mathbb{C}^2 \quad , \quad \psi_2 : D_r \longrightarrow \mathbb{C}^2$$

$$\psi_1(u) = (\underset{z_2}{u}, \underset{w_1}{u^k}) \qquad \psi_2(u) = (\underset{z_2}{-u}, \underset{w_1^2}{u^k})$$

$$w_1^2 = u^{2k} = u^d = z_2^d \qquad w_1^2 = u^{2k} = (-u)^{2k} = z_2^d.$$

We build an atlas for $\bar{R}$ where these disks are disjoint.

Case 2.



"cusp".

If d is odd:

If d is odd we can parametrize a nbd. of $(0,0)$ by an injective map from a disk. Let $u \in D$. ϕ mapping into (z_2, w_1) space.

$$\phi(u) = (u^2, u^d) \qquad z_2 = u^2 \qquad w_1 = u^d$$

satisfies $w_1^2 = u^{2d} = z_2^d$.

Check surjectivity. For each z_2 there should be 2 values of w_1. These correspond to $\pm\sqrt{z_2}$. Since d is odd $(\pm\sqrt{z_2})^d = \pm(\sqrt{z_2})^d$.

Conclusion. To a polynomial P of deg d with distinct zeros we can associate an algebraic variety $R^+ \subset \mathbb{C}^\infty \times \mathbb{C}^\infty$.

R^+ has a unique singular point

at (∞, ∞) which is a node if d is even and a cusp if d is odd. We can also associate a Riemann surface $\bar{R}$ of genus $g = \left[\frac{d+1}{2}\right]$.

We have a holomorphic map from $\bar{R} \to R^+ \subset \mathbb{C}^\infty \times \mathbb{C}^\infty$.

In particular if $d = 3$ or 4 $\bar{R}$ is a torus.

At this point we have two different ways to produce families of Riemann surfaces of genus 1 (homeomorphic to the torus).

Elliptic curves $= \{(z,w) : w^2 = P(z)\}$ and

$\mathbb{C}/\Lambda \quad \Lambda = \{m\lambda_1 + n\lambda_2 : m,n \in \mathbb{Z}\} \qquad \lambda_1, \lambda_2 \in \mathbb{C}$

λ_1, λ_2 linearly independent over $\mathbb{R}$.

Can we relate these two constructions?

Are they producing families of holomorphically (or conformally) equivalent curves? In order to address this we want to focus on conformal (holomorphic) invariants of our surface. An important invariant is the collection of meromorphic functions (and meromorphic 1-forms) on our surface.

We start by studying meromorphic functions on $\mathbb{C}/\Lambda$ where Λ is a lattice in $\mathbb{C}$.

If $g : \mathbb{C}/\Lambda \longrightarrow \mathbb{C}_\infty$ is meromorphic then the composition $f = g \circ \pi$ $\mathbb{C} \xrightarrow{\pi} \mathbb{C}/\Lambda \xrightarrow{g} \mathbb{C}_\infty$ is a meromorphic function on $\mathbb{C}$ which is invariant under the action of Λ.

Recall that that a non-constant meromorphic function on a compact Riemann surface must have at least one pole since it cannot be holomorphic. If it has just one pole then it must be a pole of order at least 2.

A formal way of constructing a doubly periodic function with just one pole on $\mathbb{C}/\Lambda$ is to consider $\mathcal{E}_\ell(z) = \sum_{\lambda\in\Lambda} (z-\lambda)^{-\ell}$ for $\ell \geq 2$.

If we can make sense of this sum then the result should be Λ invariant and have a pole of order ℓ on the points in Λ.

Step 1 is to restrict our attention to a large disk $D_R = \{|z| < R\}$. $\mathcal{E}_\ell$ will have some poles in D_R and we deal with these separately.

Let $\Lambda_R = \{\lambda\in\Lambda : |\lambda| < R\}$.

Write

$$\sum_{\lambda} (z-\lambda)^{-\ell} = \sum_{\lambda\in\Lambda_R} (z-\lambda)^{-\ell} + \sum_{\lambda\in\Lambda-\Lambda_R} (z-\lambda)^{-\ell}$$

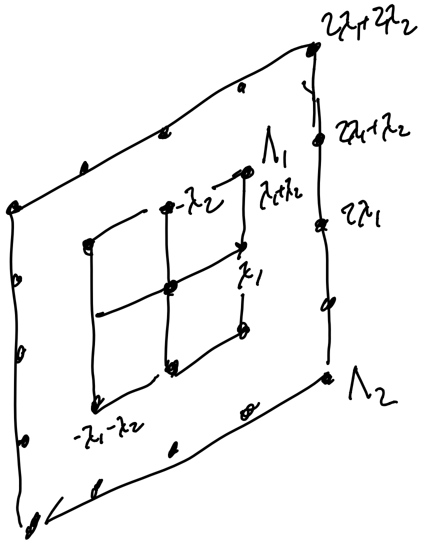
The first term makes sense and gives a meromorphic function on D_R.

We want to show that the remaining series converges inside D_R to a holomorphic function. We can try the following:

Theorem. (Weierstrass M-test). Let $W \subset \mathbb{C}$, let $f_n : W \to \mathbb{C}$ be a sequence of holomorphic functions on W. Suppose there is a sequence of positive real numbers M_n such that $|f_n(z)| \le M_n$ and $\sum M_n < \infty$ then $\sum_n f_n(z)$ converges uniformly to a holomorphic function f and $f'(z) = \sum_n f_n'(z)$.

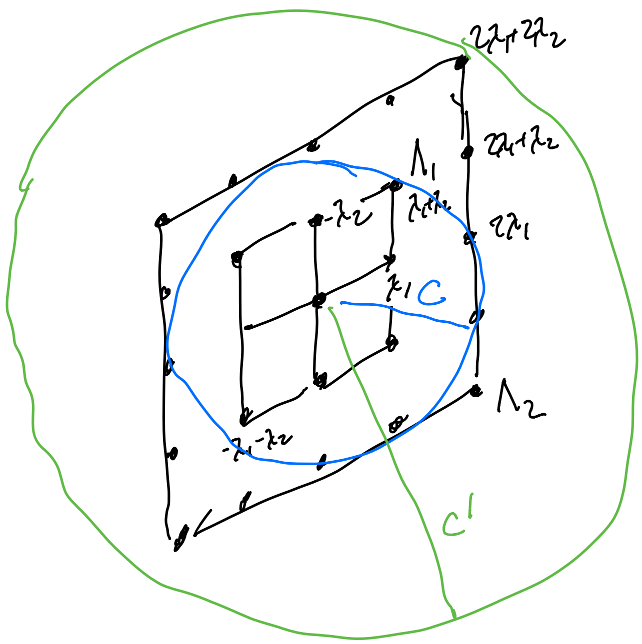
Let $\Lambda_k = \{ m\lambda_1 + n\lambda_2 : \max\{m, n\} = k \}$

Formally we can write $\sum_{\lambda \in \Lambda} (z-\lambda)^{-\ell} = \sum_{k \ge 0} \sum_{\lambda \in \Lambda_k} \frac{1}{(z-\lambda)^{\ell}}$.



Let P_k be the convex hull of Λ_k.

$$P_k = \{ r\lambda_1 + s\lambda_2 : r, s \in \mathbb{R},\ \max\{r, s\} \le k \}$$



Let c be the radius of the largest disk contained in P_1 and c' be the radius of the largest disk containing P_1.

So

$$D_c \subset P_1 \subset D_{c'}.$$

Since things scale linearly we have:

$$D_{k\cdot c} \subset P_k \subset D_{k\cdot c'}$$

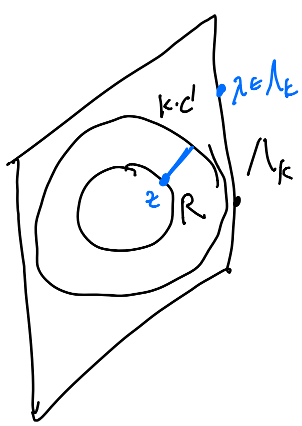
In order to apply the M-test we want to estimate the size of the term $\sum_{\lambda\in\Lambda_k} \frac{1}{(z-\lambda)^\ell}$.

Choose k_0 so that $P_{k_0} \supset D_{2R}$. For $k \geq k_0$ and $\lambda \in \Lambda_k$ if $z \in D(0,R)$ then

$d(z,\lambda) \geq R$ so $|z-\lambda| \geq k \cdot C' - R$ $\quad |z-\lambda|^{-\ell} \leq (k \cdot C' - R)^{-\ell}$

The # of lattice points in Λ_k is $4(k-1)$.

$$\left| \sum_{\lambda \in \Lambda_k} \frac{1}{(z-\lambda)^\ell} \right| \leq \frac{4(k-1)}{|k \cdot C' - R|^\ell} \leq \frac{C_0}{k^{\ell-1}} \text{ for some } C_0.$$



This is summable if $\ell - 1 \geq 2$ or $\ell \geq 3$.

Prop. For $\ell \geq 3$ the formula for E_ℓ gives a well defined meromorphic function on $\mathbb{C}$ with poles of order 3 at the points of Λ.