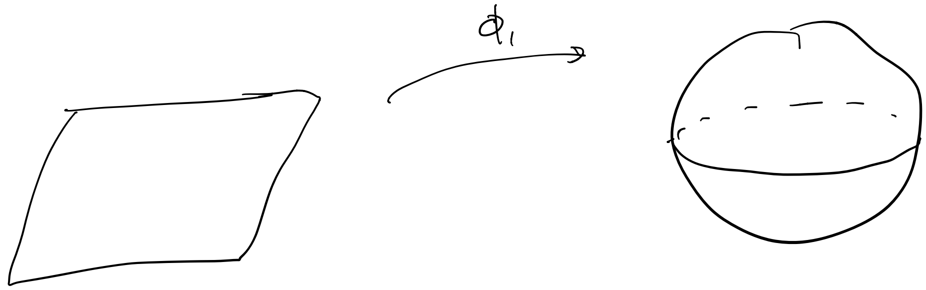
In the sphere example we could either look



$$ds = \frac{2}{1+|z|^2}|dz|$$

at our model in $\mathbb{R}^3$ to understand the geometry or look at the formula for the metric in $\mathbb{C}$.

In the case of the hyperbolic disk there is no model in $\mathbb{R}^3$. In order to understand the geometry we have to build it up from the formula.

Definition. The hyperbolic metric on the disk Δ is given by $ds = \underbrace{\frac{2}{1-|z|^2}}_{\rho(z)} |dz|$.

(Important sign change.)

$|(z,\dot z)|_h = \rho(z)\cdot|\dot z|$.

Hyperbolic space - model for negative curvature

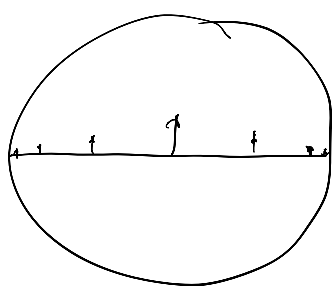
$E=F=\rho^2$ Poincaré

$G=0$.

Course graining.

$ds^2=\rho^2|dz|^2$

$$\rho(z)=\frac{2}{1-|z|^2}.$$



Unit vectors get shorter (in the picture) as we go towards the boundary of the disk.

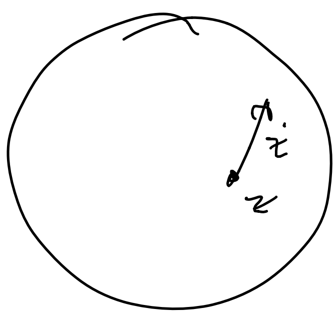
picture of unit vectors

They get shorter as you go to the edge.

A consequence of these calculations is that a linear fractional transformation preserves the hyperbolic metric on Δ.

For $z \in \Delta$ let $\quad \delta(z) = \dfrac{2}{1-|z|^2}$.

We have a metric on Δ so that the length of $(z, \dot{z})$ is $\delta(z) \cdot |\dot{z}|$. $\quad \delta(z) \cdot |dz|$.



Claim. $f \in \mathcal{D}$ preserves this metric.

Applying f to $(z, \dot{z})$ gives $(f(z), f'(z) \cdot \dot{z})$

Compare the length of a tangent vector with the length of its image.

Let's calculate $f'(z)$.

$$f'(z) = \left(\frac{az+b}{cz+d}\right)' = \frac{a(cz+d) - c(az+b)}{(cz+d)^2}$$

$$= \frac{\cancel{acz} + ad - \cancel{acz} - bc}{(cz+d)^2}$$

$$= \frac{ad-bc}{(cz+d)^2}.$$

Applied to $\begin{pmatrix} a & \bar{c} \\ c & \bar{a} \end{pmatrix}$ we get $\frac{1}{(cz+\bar{a})^2}$ factor that shows up in equation *.

$d = \bar{a}$

$c = c$

Combine with equation (*).

$$(*) \quad 1 - |f(z)|^2 = (1-|z|^2) \cdot \frac{1}{|cz+\bar{a}|^2}$$

so $|f'(z)| = \frac{1-|z|^2}{1-|f(z)|^2}$.

since:

$$\boxed{1 - |f(z)|^2 = (1-|z|^2) \cdot \frac{1}{|cz+\bar{a}|^2}}$$

This formula has geometric consequences.

Proposition. $f \in \mathrm{Aut}(\mathbb{D})$ preserves the hyperbolic metric.

Proof. To show:

$$\left|(z, \dot z)\right|_{hyp} = \left|(f(z), f'(z)\cdot \dot z)\right|_{hyp}$$

$$\left|(z, \dot z)\right|_{h} = \delta(z)\cdot|\dot z|$$

$$\left|(f(z), f'(z)\cdot \dot z)\right|_{h} = \delta(f(z))\cdot|f'(z)|\cdot|\dot z|$$

Ratio is $\dfrac{\delta(f(z))\cdot|f'(z)|}{\delta(z)}$.

Want to show:

$$|f'(z)| = \frac{\delta(z)}{\delta(f(z))}$$

But

$$\frac{\delta(z)}{\delta(f(z))} = \frac{\dfrac{2}{1-|z|^2}}{\dfrac{2}{1-|f(z)|^2}} = \frac{1-|f(z)|^2}{1-|z|^2} \overset{!}{=} |f'(z)|.$$

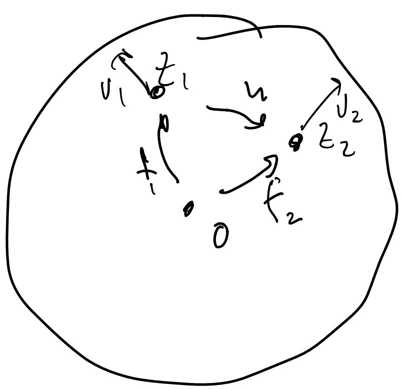
Note that any choice of a constant for δ gives an invariant metric.

Transitivity of the action implies curvature is constant.

The 2 yields a metric with curvature 1.

Thm. (Schwarz - Pick). Let h be any holomorphic map $h: \Delta \to \Delta$. Then h does not increase the hyperbolic distance.

Proof. Say $h(w_1) = w_2$.



$f_1(0) = z_1$ $\quad f_2(0) = z_2$ $\qquad f_0, f_1 \in \Delta$,

Let v_1 be a tangent vector at z_1

Let $v_2 = Dh(v_1)$. Want to show $|v_1|_h \geq |v_2|_h$

Let v_0 be the tang. vector at 0 mapping to v_1:

$Df_1(v_0) = v_1 \qquad |v_0|_h = |v_1|_h$.

Let v_3 be the tangent vector at 0 mapping to $Dh(v_1) = v_2$

$$Df_2(v_3) = v_2 \qquad |v_3|_h = |v_2|_h.$$

Suffices to show that $|v_0|_h \geq |v_3|_h$.

Consider the map $f_2^{-1} \circ h \circ f_1$.

$$\begin{aligned} D(f_2^{-1} \circ h \circ f_1)\, v_0 &= D(f_2)^{-1} \circ Dh \circ Df_1 (v_0) \\ &= D(f_2)^{-1} \circ Dh(v_1) \\ &= D(f_2)^{-1}(v_2) \\ &= v_3 \end{aligned}$$

This map is holomorphic and takes 0 to 0. By the Schwarz theorem $|v_0|_h \geq |v_3|_h$ as was to be shown.

Theorem. Say that R is a Riemann surface and $\tilde{R}$ is conformally equivalent to Δ. Then R has a metric of constant curvature -1. In this case we say R is hyperbolic.

A holomorphic equivalence between hyperbolic surfaces is an isometry of the hyperbolic metric.

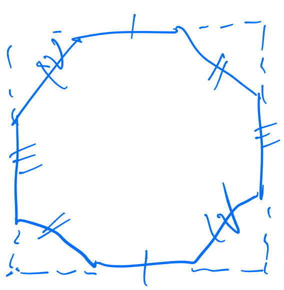
Proof. $\tilde{R} = \Delta$ and $R = \tilde{R}/\Gamma = \Delta/\Gamma$ where Γ acts by deck transformations. In particular $\Gamma \subset \mathrm{Aut}(\Delta)$ so Γ preserves the hyperbolic metric on Δ and there is a well defined metric on the quotient.

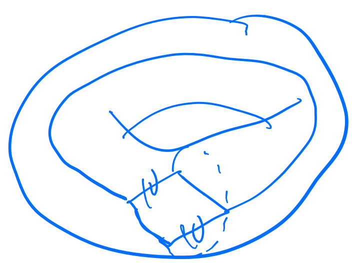
If R, S are hyperbolic and hol. equivalent then we have $f: R \to S$ holomorphic,

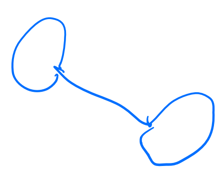
$$\begin{array}{ccc} \tilde{R} & \xrightarrow{\tilde{f}} & \tilde{S} \\ \downarrow \pi & & \downarrow \pi \\ R & \xrightarrow{f} & S \end{array}$$

$\tilde{f} \in \mathrm{Aut}(\Delta)$ so $\tilde{f}$ preserves the metric.

In general it is hard to see when two holomorphic surfaces are holomorphically equivalent while it is not so hard to decide when two surfaces are isometric.







Example.