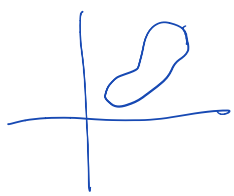
$V = \{(z,w) : w^2 = P(z)\}.$ $\qquad P(z) = c\,(z-v_1)\cdots(z-v_n)$

When V is a Riemann surface, P has simple zeros, our construct gives a "holomorphic curve" in $\mathbb{C}^2$ i.e. the holomorphic structure on V is induced by that on $\mathbb{C}^2$.

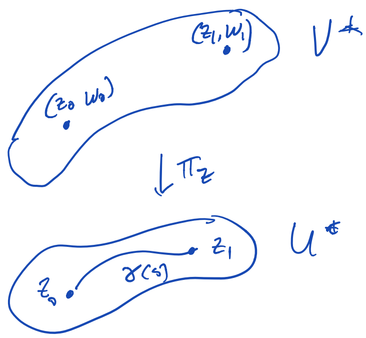


In the language of higher dimensional holomorphic maps we could say that the inclusion from V to $\mathbb{C}^2$ is holomorphic.

Equivalently the coordinate functions $\pi_z, \pi_w : \mathbb{C}^2 \to \mathbb{C}$ induce holomorphic maps on V.

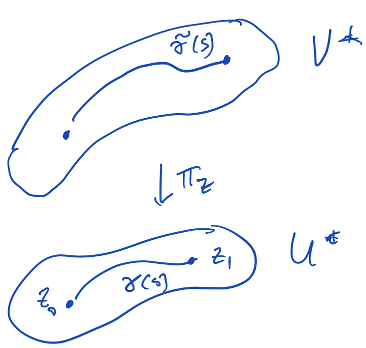
For the moment V can be singular.

Definition of $h(\gamma)$



$$h(\gamma) = \exp\left(\frac{1}{2}\int_\gamma \frac{P'(z)}{P(z)}\,dz\right).$$

Prop. If $(z_0, w_0) \in V^*$ and γ is a path in U^* from z_0 to z_1 then $(z_1, w_1) \in V^*$ where $w_1 = h(\gamma)\cdot w_0$.



Described this as lifting a path. Really we are lifting the endpoint.

A path $\gamma(s) : [0,1] \to U^*$ determines a path of paths.

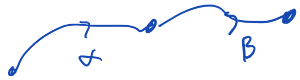
$$\gamma_s(t) = \gamma(t) : [0, s] \to U^*$$

$$\tilde{\gamma}(s) = (\gamma(s), h(\gamma_s))$$

$$\tilde{\gamma}(s) = \left(\gamma(s), \int_0^s \frac{P'(\gamma(t))}{P(\gamma(t))}\gamma'(t)\,dt\right)$$

formula for lifted curve. Same proof shows $\tilde{\gamma}(s) \in V$.

Properties of $h(\gamma)$.

If 

α and β are paths with

$\alpha(1) = \beta(0)$ then $h(\alpha\cdot\beta) = h(\alpha)\cdot h(\beta)$.

Proof. We have additivity for path integrals. Since we are exponentiating we get multiplicativity.

If γ is a loop in U^* then $h(\gamma) = \pm 1$.

Proof $h^2(\gamma) = \dfrac{P(\gamma(1))}{P(\gamma(0))} = 1$. So $h(\gamma) = \pm 1$.

These two observations give:

h induces a homomorphism from $\pi_1(U^*, z_0)$ to the multiplicative group $\{\pm 1\}$.

We would like a formula for this homomorphism.

Prop. Say $P(z) = c \cdot \prod (z - r_j)$

If γ is a loop then $\int_\gamma \frac{P'(z)}{P(z)} dz =$

$= 2\pi i \sum \mathrm{wind}(\gamma, r_j)$.

Proof. (Recall that log turns a product into a sum. We will look at the analogous differential condition.)

$$\int_\gamma \frac{P'(z)}{P(z)} dz = \int_\gamma \frac{(c \prod (z - r_j))'}{c \prod (z - r_j)} dz$$

$$= \int_\gamma \sum_k \frac{c(z-z_1) \cdots \overbrace{(z-z_k)} \cdots (z-z_j)}{c(z-z_1) \cdots (z-z_j)}$$

$$= \int_\gamma \sum_k \frac{dz}{z-z_j} dz$$

$$= 2\pi i \sum_k \mathrm{wind}(\gamma, z_j)$$

$$\int_\gamma \frac{(PQ)'}{PQ} dz$$

$$= \int_\gamma \frac{P'Q + PQ'}{PQ} dz$$

$$= \int_\gamma \frac{P'}{P} + \frac{Q'}{Q} dz = \int_\gamma \frac{P'}{P} dz + \int_\gamma \frac{Q'}{Q} dz$$

Apply repeatedly to a poly the poly $P = c \prod (z - r_j)$ to get $\int_\gamma \frac{P'}{P} dz = \int_\gamma \sum_k \frac{dz}{z - r_j} dz$.

Prop. $h(\gamma) = (-1)^{\frac{1}{2}\sum \mathrm{wind}(\gamma, v_j)}$

Proof. $h(\gamma) = \exp\left(\frac{2\pi i}{2} \sum \mathrm{wind}\right)$

$= \exp\left(\pi i \sum \mathrm{wind}\right)$

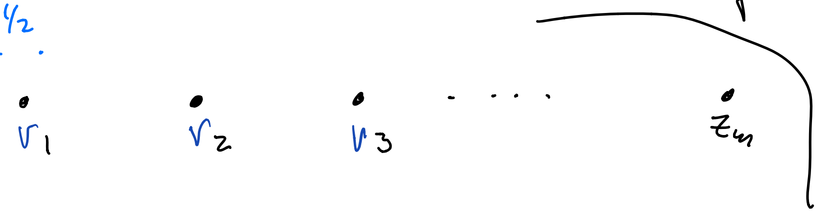
$= \exp(\pi i)^{\sum \mathrm{wind}}$

$= (-1)^{\sum \mathrm{wind}}.$

Procedure is to cut V into pieces that we can understand and then calculate how to glue the pieces back together.

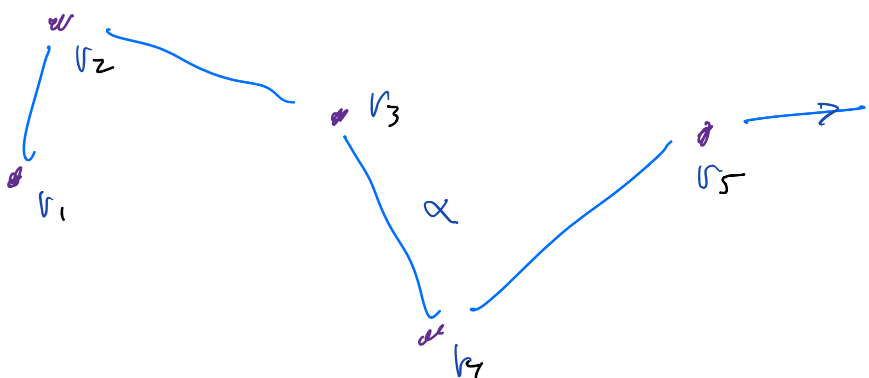
$P(z) = \sqrt{(z-r_1)(z-r_2)\cdots(z-r_m)}$

$z^{1/2}$.



$|f(z)|$ is "single valued". Does not depend on the branch chosen.

Choose a path $\alpha : \mathbb{R}^+ \to \mathbb{C}$ which passes through all of the roots and so that $\alpha(s) \to \infty$ as $s \to \infty$.



Let $\hat{U} = \mathbb{C}$ - image of α.

Claim that $\hat{U}$ is evenly covered.

If $\pi : V^* \to U^*$ is the covering map then $\hat{U} \subset U^*$ and $\pi^{-1}(\hat{U})$ is a disjoint union of V_0 and V_1.

a priori there are two "proofs".

Could claim that $\hat{U}$ is a topological disk and hence simply connected or (in this case)

we can give a more honest proof and more explicit proof.

Prop. We can construct 2 disjoint branches of $\sqrt{P(z)}$ over $\hat{U}$ by constructing to

Proof. Pick a base point z_0 in $\hat{U}$.

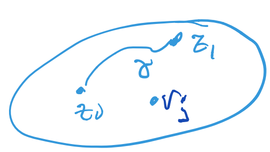
$P(z_0) \neq 0$. Let w_0 and w_0' be the two solutions of $w^2 = P(z_0)$.

$w_0' = -w_0$

$q = (z_0, w_0)$ $q' = (z_0, w_0')$ be the two points in $\hat{V}$ that project to z_0 under $\pi_z : \hat{V} \to \hat{U}$.

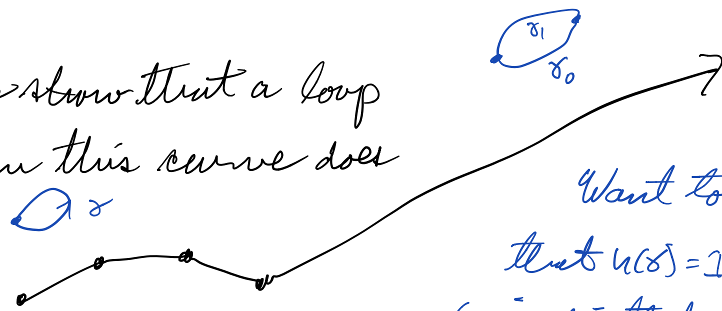
Define functions $\psi, \psi' : \hat{U} \to \hat{V}$. Define $\psi(z_1)$

$$\psi(z_1) = (z_1, w_0 h(\gamma)) \quad \text{where } \gamma \text{ is a path}$$
$$\psi'(z_1) = (z_1, \underbrace{-w_0 h(\gamma)}_{w_0' h(\gamma)}) \quad \text{from } z_0 \text{ to } z_1$$



To see that ψ, ψ' are well defined we need to check $h(\gamma)$ is independent of γ.

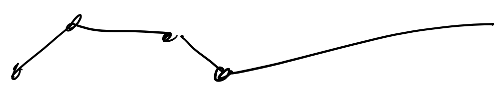
Want to show that a loop disjoint from this curve does



not go around any point in the curve.

Relies essentially on the curve going to ∞.

Want to show that $h(\gamma)=1$. This implies that we can build a square root of P i.e. the complement of the curve is evenly covered.



Not true for a curve that stops at ~~some~~ point.

We use $h(\gamma) = (-1)^{\sum \mathrm{wind}(\gamma, r_j)}$.

Show that $\mathrm{wind}(\gamma, r_j) = 0$ for all j.

$$\mathrm{wind}(\gamma, r_j) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - r_j} = 0$$

Let $\alpha(s)$ be the path containing r_j disjoint from γ with $\alpha(s) \to \infty$. $\mathrm{wind}(\gamma, \alpha(t))$ constant in t

By continuity and disjointness $\alpha(t)$ is constant in t. (continuous $\mathbb{Z}$ valued function). But $\frac{1}{z-\alpha(s)}$ goes to 0 uniformly in z so the integral tends too for every $\gamma(t)$

$$\int_\gamma \frac{dz}{\gamma - \alpha(s)} = \int_0^1 \underbrace{\frac{\gamma'(t)}{\gamma(t)-\alpha(s)}} dt$$

goes to 0 uniformly in s for each $t \in [0,1]$.