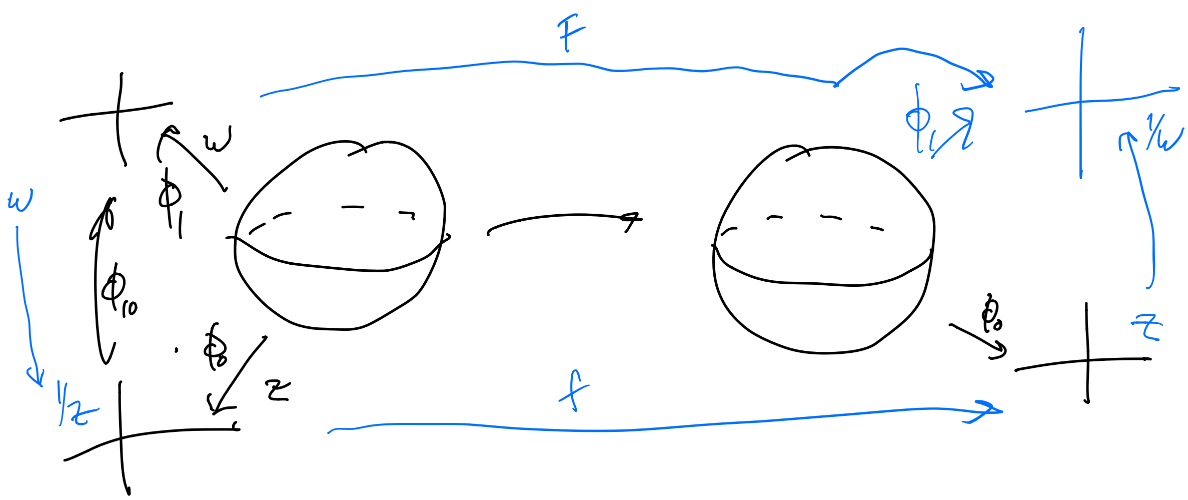
Automorphisms of the complex plane.

Theorem. A holomorphic function $f: \mathbb{C} \to \mathbb{C}$ is in $\mathrm{Aut}(\mathbb{C})$ if and only if $f(z) = az + b$ for some constants $a \neq 0$ and b.

Proof. Say that $f: \mathbb{C} \to \mathbb{C}$ is holomorphic and injective. We can view f as a hol. map $\mathbb{C} \xrightarrow{f} \mathbb{C}_\infty$.

Is it possible to extend f to a holomorphic function whose domain is $\mathbb{C}_\infty$?



$F(w) = \frac{1}{f(1/z)}$. F is holomorphic in $\mathbb{C}-\{0\}$.

F has an isolated singularity. What kind of singularity is it? Any $w_j \to 0$. Let $z_j = \frac{1}{w_j}$.

If $z_j \to \infty$ then z_j eventually leaves all compact sets in $\mathbb{C}$. Since f is a homeomorphism $f(z_j)$ eventually leaves all compact sets in $\mathbb{C}$ so $f(z_j) \to \infty$. Thus $F(w_j) \to 0$. So F has a removable singularity. f has a holomorphic extension. $f(z) = \frac{az+b}{cz+d}$. $f(\infty)=\infty$ so $\frac{a}{c}=\infty$ $c=0$

$f(z) = \left(\frac{a}{d}\right) + \frac{c}{d}$.

New Topic.

Say $f: R \to S$ is a hol. map between Riem. surfaces. For $w \in S$ we can consider

$\deg(f) = \# \{z \in R : f(z) = w\}$? Some version of the degree of the map?

This function is not well behaved. Take

$R = S = \mathbb{C}$ $\quad f(z) = z^n$ for $n > 1$. $w = 0$ in S has 1 inverse image while $w \neq 0$ has n inverse images.

We know how to fix this. $v_f(0)$ is the multiplicity of $0 \in R$ as a solution to $f(z) = 0$. We define

$$\delta(w) = \sum_{\substack{z \in R \\ f(z) = z_0}} v_f(z).$$

Counts # of inv. images but gives extra weight to some.

$v_f(z) \geq 1$

In our example $v_f(0) = n$ so $\delta(w)$ is independent of w.

Is this true for any map?

No. Let $R = \mathbb{C} - \{0\}$, $S = \mathbb{C}$. $f(z) = z^m$ For $0 \in S$ $\delta(0) = 0$ while $\delta(w) = m$ for any other w.

Thm. If R is compact, f is not constant and S is connected then for $z_0 \in S$ $\delta(z_0)$ is constant.

Def. We call the common value of δ the degree of the map.

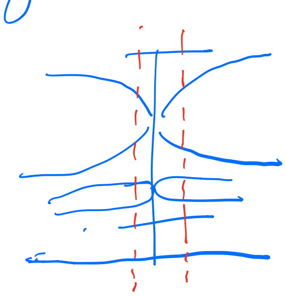
Proof. If we pick $w_0 \in S$ then the solutions of $f(z) = w_0$ are isolated so by compactness there are only finitely many. Call these $z_1 \dots z_n$. Our theorem on the local form of holomorphic maps tells us that we can choose a nbd U_j and chart $\phi_j : U_j \to \Delta$ and a chart ψ_0 around w_0 so that $\psi_0 : V_0 \to \Delta$

$$\begin{tikzcd} U_j \arrow[r, "f"] \arrow[d, "\phi_j"] & V_0 \arrow[d, "\psi_0"] \\ \Delta_{r_j} \arrow[r, "z \mapsto z^{n_j}"'] & \Delta_{r_j} \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} U_j \arrow[r, "f"] \arrow[d, "\phi_j"] & V_0 \arrow[d, "\psi_0"] \\ \underset{z}{\Delta_r} \arrow[r, "w = z^{n_j}"'] & \underset{w}{\Delta_r} \end{tikzcd}$$

where $n_j = \nu_f(z_j)$. Let $r = \min\{r_j\}$.

We have $S(w_0) = \sum_j \nu_f(z_j) = \sum n_j$.

Want to show that $S(w)$ is constant for w near w_0 (and use connectedness).



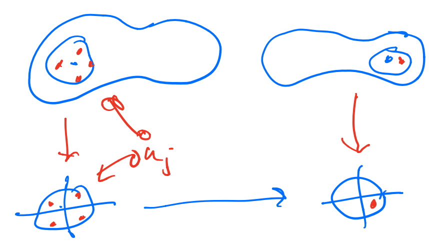
Let $E = R - \bigcup_j U_j$.

For $w^{\neq w_0}$ near w_0 we have

$$S(w) = \sum_{\substack{z_k \in R \\ f(z_k)=w}} v_f(z_k) = \sum_j \sum_{\substack{z \in U_j \\ f(z)=w}} v_f(z) + \sum_{z \in E} v_f(z).$$

Now using the local form of f in each U_j we have

$$\sum_{\substack{z \in U_j \\ f(z)=w}} v_f(z) = \sum_{\substack{z \in U_j \\ f(z)=w}} 1 = n_j$$



So the first term is $\sum_j n_j$. What about the second term? Note that it is this second term that contributes in example 2.

$E = R - \bigcup_j U_j$ is compact because it is a closed space of a compact space.

E contains no inverse images of w_0 (these are the $z_1 \dots z_j$ in the U_j).

It follows that $f(E)$ is a compact subset of S not containing w_0 so there is a nbd. of w_0 not in $f(E)$. In this nbd. there is no contribution from the last term so $S(w)$ is constant.

(We call the structure that we have established a branched covering map. Every pt. has a nbd. which is "evenly branched".)

Definition. We call $d_f \equiv d_f(q)$ the <u>degree</u> of f. Note $d_f \geq 1$. (We saw before that a non constant hol. map is surjective. This is a refinement of that.).

Compare with smooth case: degree is defined in both cases. Smooth case we can count inverse images of a generic point by Sard's thm.

Here we can count inverse images of <u>all</u> pts.
Def. of degree requires a choice of orientation in our case Riemann surfaces come with a choice of orientation. $d(f) \geq 1$ has no analogue in the general case.