Example. Let λ and μ be non-zero complex numbers which are linearly independent over $\mathbb{R}$.

Let $\Gamma = \{m\lambda + n\mu : m, n \in \mathbb{Z}\}$. Γ acts additively on $\mathbb{C}$.

$$(m\lambda + n\mu)(z) = m\lambda + n\mu + z.$$

Claim that $\mathbb{C}/\Gamma$ is a Riemann surface homeomorphic to the torus.

Write $\lambda = a+bi$ μ as $c+di$.

Identify $\mathbb{C} = x+iy$ with $\mathbb{R}^2 = \begin{pmatrix} x \\ y \end{pmatrix}$.

Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

A takes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} c \\ d \end{pmatrix}$.

Since λ, μ are lin. independent over $\mathbb{R}$,

A is invertible. A^{-1} takes $\begin{pmatrix} a \\ b \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

It converts the action of Γ to the action of $\mathbb{Z}^2$.

Thus $\mathbb{C}/\Gamma \approx \mathbb{R}^2 / m\begin{pmatrix} a \\ b \end{pmatrix} + n\begin{pmatrix} c \\ d \end{pmatrix} \overset{A^{-1}}{\approx} \mathbb{R}^2 / m\begin{pmatrix} 1 \\ 0 \end{pmatrix} + n\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbb{R}^2/\mathbb{Z}^2 = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$
$= S^1 \times S^1.$

4.1 ANALYTIC FUNCTIONS

Let R and S be Riemann surfaces with atlases $\{(\phi_\alpha, U_\alpha)\}$, $\{(\psi_\beta, V_\beta)\}$ respectively. Any function $f : R \to S$ can be expressed locally in terms of local coordinates by the functions

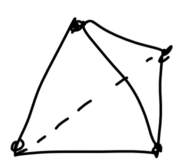
$$f_{\beta\alpha} = \psi_\beta \, f \, (\phi_\alpha)^{-1} \, . \qquad (4.1.1)$$

Note that $f_{\beta\alpha}$ is defined on the subset $\phi_\alpha(U_\alpha \cap f^{-1}(V_\beta))$ of $\mathbb{C}$: if f is continuous, this set is open.

__Definition__ 4.1.1. Let R and S be Riemann surfaces. A continuous function $f : R \to S$ is ~~analytic~~ holomorphic on R if and only if each $f_{\beta\alpha}$ is holomorphic.

Def. We say that two Riemann surfaces R, S are holomorphically equivalent if there is a holomorphic bijection $f: R \to S$ with a holomorphic inverse.

Example: $\mathbb{CP}^1$, $\mathbb{C}_\infty$ and S^2 are all holomorphically equivalent.

Example?: S^2 and  the tetrahedron

are homeomorphic but not clearly

holomorphically equivalent or clearly inequivalent.

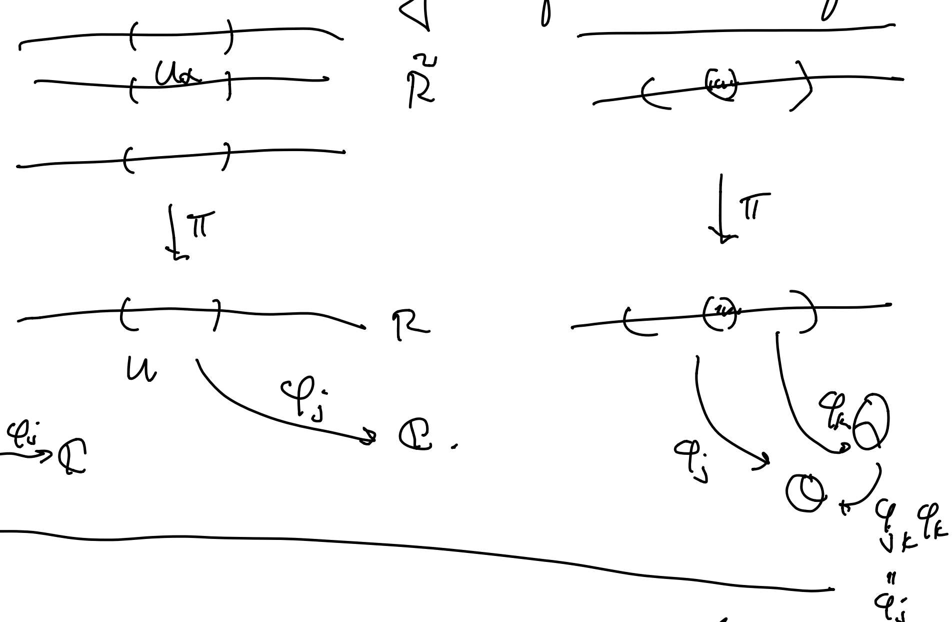
Example: The unit disk D_1 and $\mathbb{C}$ are homeomorphic but <u>not</u> holomorphically equivalent.

Proof. If they were holomorphically equivalent there would be a holomorphic function $f : \mathbb{C} \to D_1$. Viewed as a function from

$\mathbb{C} \to D_1 \subset \mathbb{C}$ $\quad$ $\mathbb{C}$ to $\mathbb{C}$ f would be bounded but not constant. This violates Liouville's Theorem that a bounded holomorphic function on $\mathbb{C}$ is constant.

Proposition. Let R be a Riemann surface. Let $\tilde{R}$ be a covering space of R. Then $\tilde{R}$ has a natural Riemann surface structure for which the covering map is holomorphic.

Proof.



$U_\alpha \xrightarrow{\pi} U \xrightarrow{\varphi_i} \mathbb{C}$

Thm. Every Riemann surface R can be presented as a simply connected Riemann surface modulo the action of a group of holomorphic automorphisms.

Proof. Let $\tilde{R}$ be the universal covering surface of R. Let Γ be the deck group

Course objective:

Main Theorem. Every simply connected Hausdorff Riemann surface is conformally equivalent to $\mathbb{D}$, $\mathbb{C}$ or S^2.

Cor. Every Riemann surface has the form S^2/Γ, $\mathbb{C}/\Gamma$ or $\mathbb{D}/\Gamma$ where Γ is acting holomorphically.

$\downarrow$
S^2

There is a uniform construction of all Riemann surfaces

Cor. Every Hausdorff Riemann surface is 2nd countable.

From now on we assume that all Riemann surfaces are Hausdorff.

Lecture ends here