Let's check that $\mathcal{E}_\ell$ is translation invariant for $\ell \geq 3$.

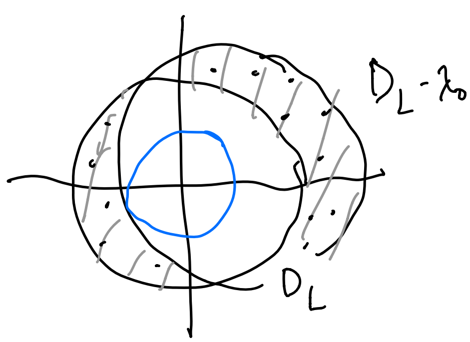
Fix $R > 0$. Let $\lambda_0 \in \Lambda$. Want to show that $\mathcal{E}_\ell(z+\lambda_0) = \mathcal{E}_\ell(z)$

For $z \in D_R$ we can find an $L > 2R$ write

$$\left| \mathcal{E}_\ell(z) - \sum_{\lambda \in D_L \cap \Lambda} (z-\lambda)^{-\ell} \right| \leq \varepsilon \quad \text{and}$$

$$\left| \mathcal{E}_\ell(z+\lambda_0) - \sum_{\lambda \in D_L \cap \Lambda} (z+\lambda_0-\lambda)^{-\ell} \right| = \left| \mathcal{E}_\ell(z+\lambda_0) - \sum_{\lambda \in D_L - \lambda_0} (z-\lambda)^{-\ell} \right| \leq \varepsilon$$

So $$\left| \mathcal{E}_\ell(z+\lambda_0) - \mathcal{E}_\ell(z) \right| \leq \underbrace{\left| \sum_{\lambda \in D_L \triangle (D_L - \lambda_0) \cap \Lambda} (z-\lambda)^{-\ell} \right|}_{\text{Bounded by the tail of } \sum_{j \geq J} \omega_j \text{ where } J \to \infty \text{ as } L \to \infty} + 2\varepsilon$$




Bounded by the tail of $\sum_{j \geq J} \omega_j$ where $J \to \infty$ as $L \to \infty$.

Letting $\varepsilon \to 0$ causes $L \to 0$ and we see that $\mathcal{E}_\ell(z+\lambda_0) = \mathcal{E}_\ell(z)$ for $z \in D_R$. Let $R \to \infty$.

What about constructing a meromorphic function of degree 2?

One approach would be to integrate the 1-form $E_3 dz$. Presumably would produce something of the form $\sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^2} + c_\lambda$.

Instead we will simply pick constants $c_\lambda$ that work. Consider

$$\frac{1}{z^2} + \sum_{\lambda \in \Lambda - \{0\}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}$$

Then $$\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} = \frac{\lambda^2 - (z-\lambda)^2}{\lambda^2 (z-\lambda)^2} = \frac{\lambda^2 - z^2 + 2\lambda z - \lambda^2}{\lambda^2 (z-\lambda)^2}$$

$$= \frac{z(2\lambda - z)}{\lambda^2 (z-\lambda)^2}$$

where $|z|$ is bounded. $\left| \frac{z(2\lambda - z)}{\lambda^2 (z-\lambda)^2} \right| \le C_0 \frac{|\lambda|}{|\lambda^4|} = \frac{C_0}{|\lambda^3|}$.

Following the analysis we did before, recalling that the # of terms grows linearly we get $M_k \le \frac{C_1}{k^2}$. So we get convergence.

Def $\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}$.

Is $\wp(z)$ invariant?

Note $\wp'(z) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3} = \mathcal{E}_3(z)$.

So $\wp'(z+\lambda) = \wp'(z)$ and $\wp(z+\lambda) = \wp(z) + c_\lambda$

In particular $\frac{d}{dz}\left(\wp(z+\lambda) - \wp(z)\right) = 0$ so

$\wp(z+\lambda) = \wp(z) + c$.

Note also that $\wp(z) = \wp(-z)$ since

$$\wp(z) = \frac{1}{z^2} + \sum_{k} \sum_{\lambda \in \Lambda_k} (z-\lambda)^{-2} - \lambda^{-2}$$

$$\wp(-z) = \frac{1}{z^2} + \sum_{k} \underbrace{\sum_{\lambda \in \Lambda_k} (-z-\lambda)^{-2} - \lambda^{-2}}$$

$$= \sum_{-\lambda \in \Lambda_k} (-z+\lambda)^{-2} - \lambda^{-2} = \sum_{-\lambda \in \Lambda_k} (z-\lambda)^{-2} - \lambda^{-2}.$$

Prop. $\wp(z) = \wp(z+\lambda)$.

Let $\Gamma$ be the group generated by $z \mapsto -z$ and $z \mapsto z+\lambda$ for $\lambda \in \Lambda$.

Elements of this group act with fixed points: $z \mapsto -z+\lambda$ fixes $z = -z+\lambda$ $2z = \lambda$, $z = \frac{\lambda}{2}$.

If $\lambda_0 \in \Lambda - 2\Lambda$ and $z_0 = \lambda_0/2$ then $z_0$ is fixed by $z \mapsto -z + \lambda_0$ but $z_0 \notin \Lambda$ so $z_0$ is not a pole of $P$. We have $P(z_0) = P(-z_0)$

$= P(-z_0 + \lambda_0) + C_{\lambda_0} = P(z_0) + C_{\lambda_0}$ so $C_{\lambda_0} = 0$.

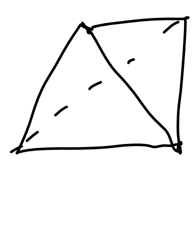
Similarly $C_{\lambda_1} = 0$. $P(z_0 + \lambda + \lambda') = P(z_0 + \lambda) + C_{\lambda'}$

$= P(z_0) + C_\lambda + C_{\lambda'}$ so $\lambda \mapsto C_\lambda$ is a homomorphism. Since it vanishes on generators of $\Lambda$ it vanishes on all of $\Lambda$.

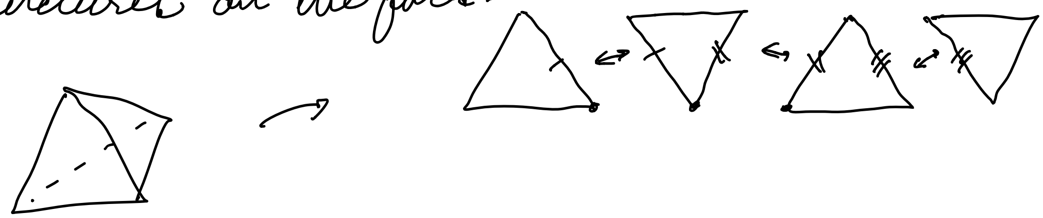
Thus $P$ is periodic and induces a meromorphic function on $\mathbb{C}/\Lambda$ of degree 2.

$P$ is called the Weierstrass $P$ function.

Recall our construction of Riemann surface structures on polygonal regions in $\mathbb{R}^3$.

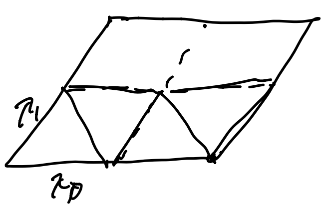


These regions have well defined metrics on their faces. These metrics together with the choice of an orientation give conformal structures on the faces.

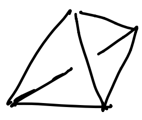


We can also think of $\mathbb{C}/\Lambda$ as having not just a conformal structure but a metric structure. $\Gamma$ action preserves this metric structure.

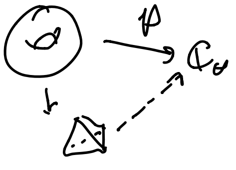
Example.



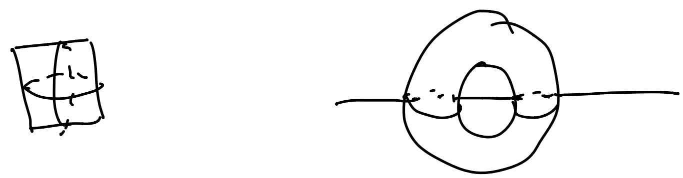

Take $\lambda_0 = 1$, $\lambda_1 = e^{2\pi i/6}$



Quotient surface $\mathbb{C}/\Gamma$ can be identified with the boundary of the tetrahedron. We constructed a Riemann surface structure on this surface earlier in the course.



$\wp$ induces a conformal isomorphism from the boundary of the tetrahedron to $\mathbb{C}_\infty$, So this  is conf. isomorphic to $\mathbb{C}_\infty$.



This picture suggests that the points of valence 2 for $\wp$ are the half-lattice points. Let's check this.

$\wp' = -2\mathcal{E}_3$ and $\mathcal{E}_3 = \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3}$ is an odd function.

If $\lambda_0 \in \Lambda/2$ then

$$\wp'(\lambda_0) = -\wp'(-\lambda_0) = -\wp'(-\lambda_0 + 2\lambda_0) = -\wp'(\lambda_0).$$

So $\wp'(\lambda_0) = 0$. This implies that for $\lambda_0/2$, $\lambda_1/2$ and $\lambda_0+\lambda_1/2$ $\wp$ has valence at least 2. But $\wp$ has degree 2 so the valence must be exactly 2.

Since $\wp$ has a pole of order 2 at 0.

Note that this is consistent with the Riemann-Hurwitz formula:

$$\underset{\substack{\| \\ -4}}{\chi(T) - 2\chi(S^2)} = \sum_{V_p(p) \geq 1.} 1 - V_p(p) = 4(-1).$$

Prop. The values of $\wp$ at $0, \frac{\lambda_0}{2}, \frac{\lambda_1}{2}, \frac{\lambda_0+\lambda_1}{2}$ are distinct.

Prop. $(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3$ for certain constants $g_2$ and $g_3$.

Proof. $\wp(z) - \frac{1}{z^2} = \sum_{\lambda \in \Lambda - \{0\}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}$

vanishes at 0 since each term vanishes

at 0 and it is an even function

(since $\wp$ and $\frac{1}{z^2}$ are even functions).

Thus $\wp(z) = z^{-2} + \lambda z^2 + \mu z^4 + O(z^6)$

$$\wp'(z) = -2z^{-3} + 2\lambda z + 4\mu z^3 + O(z^5)$$

$$(\wp'(z))^2 = 4z^{-6} - 8\lambda z^{-2} - 16\mu + O(z^4)$$

$$\wp^3(z) = z^{-6} + 3\lambda z^{-2} + 3\mu + O(z^4)$$

$$(\wp'(z))^2 - 4\wp^3(z) = \underbrace{-20\lambda z^{-2} - 28\mu}_{-20\lambda \wp(z) - 28\mu} + O(z^3)$$

$$-20\lambda \wp(z) - 28\mu + O(z^3)$$

Thus $(\wp'(z))^2 - 4\wp^3(z) + 20\lambda\,\wp(z) + 28\mu$ has no pole at 0 and has value 0 at 0.

Since the only poles of $\wp$ and $\wp'$ occur at lattice points this function has no other poles.

Thus we have a hol. fcn on a compact Riemann surface so it is constant.

Evaluating at 0 we see that it vanishes.

Cor. The polynomial $4z^3 - 20\lambda z - 28\mu$ has distinct 0's.

Cor. The functions $z \overset{\Phi}{\mapsto} (\wp(z), \wp'(z))$ from $\mathbb{C}/\Lambda$ to $\mathbb{C}^2$ parametrize the curve $R = \{(z,w) : w^2 = 4z^3 - 20\lambda z - 28\mu\}$.

This parametrization extends to the Riemann surface $\tilde{R}$ where $0 \mapsto (\infty, \infty)$.

Remark. $\mathbb{C}^2 \to \mathbb{CP}^2$ gives an alternative compactification of $\mathbb{C}^2$ and of $R$. In degree 3 and 4 this comp. of $R$ is non-singular. In higher degree it is singular.

We can think of $\wp(z)$ and $\wp'(z)$ as being the analogues of $\sin(z)$ and $\cos(z)$ for $R$ instead of $z^2 + w^2 = 1$ in that

$\cos = \sin'$ and $\sin^2 + \cos^2 = 1$

Recall that the elliptic integral (after which the elliptic curve is named) corresponds to

$$\int \frac{dz}{\sqrt{f(z)}}.$$

On $R$ this became $\int \frac{dz}{w}$.

Prop. $\phi^*\left(\frac{dz}{w}\right) = du$

Proof. $\phi^*\left(\frac{dz}{w}\right) = \phi^*(dz) \cdot \frac{1}{\phi^*(w)}$

$$= d\,\phi^*(z) \cdot \frac{1}{\phi^*(w)}$$

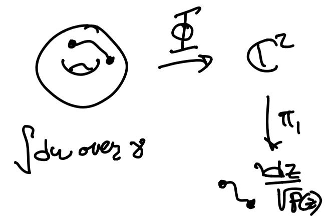
$$= \frac{d\,P(u)}{P'(u)}$$

$$= \frac{P'(u)du}{P'(u)}$$

$$= du.$$

$\Phi_1(z) = P(z).$

Cor. $\displaystyle\int_{z_0}^{z_1} \frac{dz}{\sqrt{P(z)}} = \wp^{-1}(z_1) - \wp^{-1}(z_0).$




Say we have a

$$\int_\gamma dz = \int_\gamma (\pi_1 \circ \Phi)^* \theta = \int_{\pi_1 \circ \Phi(\gamma)} \theta = \int_{\Phi_1(\gamma)} \theta = \int_{\wp(\gamma)} \theta$$

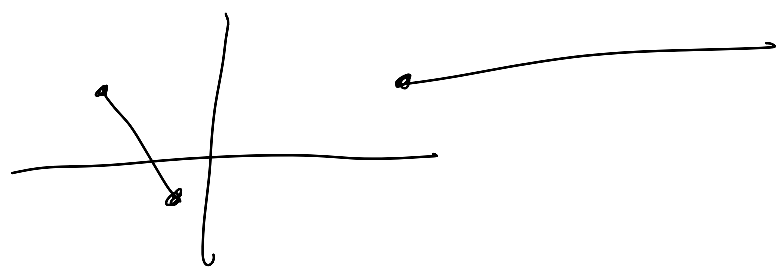
Now write $\rho = \wp(\gamma)$ and $\gamma = \wp^{-1}(\rho)$.

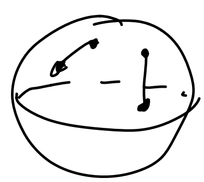
$$\wp^{-1}(1) - \wp^{-1}(0) = \int_{\wp^{-1}(\rho)} dz = \int_\rho \theta$$

$$\int_{P(\delta)}$$

makes sense when we choose a branch for $\delta$ and choose a branch for $P^{-1}$. Different branches for $P^{-1}$ differ by a constant.

Example.





Note that we can also integrate $\frac{dz}{w}$ over closed loops in $R$.

We call these integrals periods.

The collection of periods is a subgroup of $\mathbb{C}$ since $\int : \pi_1(R) \to \mathbb{C}$ is a homomorphism.

$$\gamma \mapsto \int_\gamma \frac{dz}{w}$$

$\Lambda$ is the lattice of periods.

Note that if a Riemann surface has a non-vanishing hol. 1-form then any 2 such forms differ by multiplication by a scalar.

Cor. Given an elliptic surface $R$ the collection of homomorphisms from $\pi_1(R) \to \mathbb{C}$ obtained by integrating hol. 1-forms is a 1-dimensional complex subspace of $\mathrm{Hom}(\pi_1(R), \mathbb{C})$ which has dim 2.

A characteristic property of the torus is the existence of a non-vanishing hol. 1-form. Just as the existence of a mero. fcn. with 1 pole is a char. property of the sphere.

For a general we can consider the subspace of $\mathrm{Hom}(\pi_1(R), \mathbb{C})$ corresponding to integration of hol 1-forms.

Given a lattice $\Lambda$ we have described how to find an elliptic variety. Say we have a $P$ how do we find the lattice $\Lambda$?

Step 1. Show that $R = \{w^2 = P(z)\}$ has a non-zero holomorphic 1-form.

Hol 1-form gives a system of charts coming from integration. These differ by a constant.

Cor. Given a meromorphic 1-form $\theta$ on $R$ $\quad$ $\#\text{poles}\,\theta - \#\text{zeros}\,\theta = \chi(R)$.

Follows from the Main Theorem that a Riemann surface homeomorphic to a Torus can has the form $\mathbb{C}/\Lambda$ with $\Lambda$ a lattice. In particular such a surface has a non-zero hol. 1-form.

We can identify the collection of elliptic curves with 4-tuples of distinct points in $\mathbb{CP}^1$.

Prop. Given 4 distinct points in $\mathbb{CP}^1$ there is a holomorphic map from $T^2$ to $\mathbb{CP}^1$ branched at exactly those 4 points.

Proof. $\bar{R} \to \mathbb{C}_\infty \times \mathbb{C}_\infty \xrightarrow{\pi_1} \mathbb{C}_\infty$ is branched at the 3 roots of $p$ and at $\infty$.