$V = \{(z,w) : w^2 = P(z)\}$. $\qquad P(z) = c\,(z-r_1)\cdots(z-r_4)$

r_j are distinct

basepoint in U $\qquad \hat{U} = U - \{\alpha(s)\}$

(z_0, w_0) in V

$\psi, \psi' : \hat{U} \longrightarrow V$

$\psi(z_1) = (z_1, w_0 h(\gamma))$ $\qquad \gamma$ disjoint from α

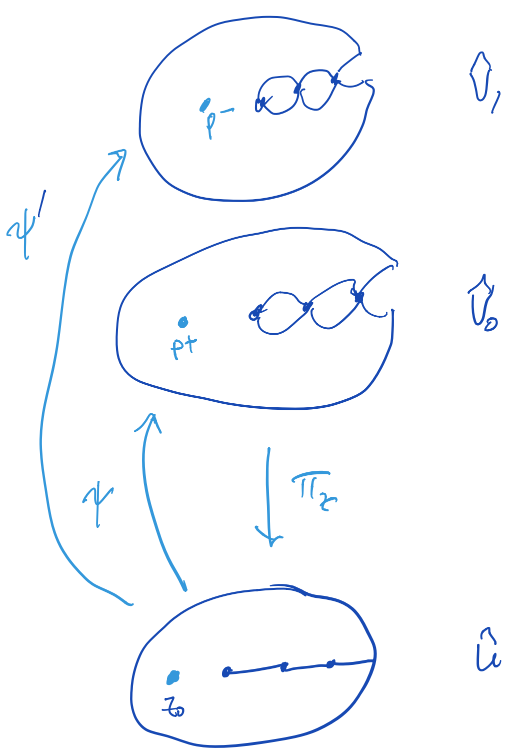
$\psi'(z_1) = (z_1, -w_0 h(\gamma))$

inverses to π_z, lifting maps

Claim. ψ, ψ' extends continuously to the r_j and takes the value $(v_j, 0)$ at r_j.

Proof. The equation $\psi^2(u) = P(u)$ gives us $|\psi(u)|^2 = |P(u)|$ so $|\psi(u)| = \sqrt{|P(u)|}$.

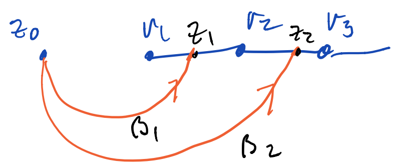
If $u \to z_j$ then $|P(u)| \to 0$ so $|\psi(u)| \to 0$.



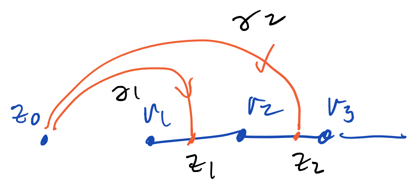
Assume that the v_j lie on the real axis.

We will define two extensions of ψ to α:
a lower extension and an upper extension.

We do these by choosing specific paths to integrate along.



lower extension of ψ_ℓ



upper extension of ψ_u

At z_j with j odd we claim that the lower extension of ψ_ℓ agrees with the upper extension of ψ'!

$$\psi_\ell(z_j) = (z_j,\ w_0\, h(\beta_j)) \qquad \text{lower ext of } \psi$$

$$\psi'_\ell(z_j) = (z_j,\ -w_0\, h(\gamma_j)) \qquad \text{upper ext of } \psi'$$

Check $w_0 \cdot h(\beta_j) = -w_0\, h(\gamma_j)$ or $h(\beta_j) = -h(\gamma_j)$

or $h(\beta_j \cdot \gamma_j^{-1}) = (-1)^{\sum \mathrm{wind}(\beta_j \cdot \gamma_j^{-1})} = -1$.

Similarly at z_j with j odd the upper extension ψ_u of ψ agrees with the lower extension of ψ'! ψ'_ℓ

When j is even the upper and lower extensions of ψ (or ψ') agree.

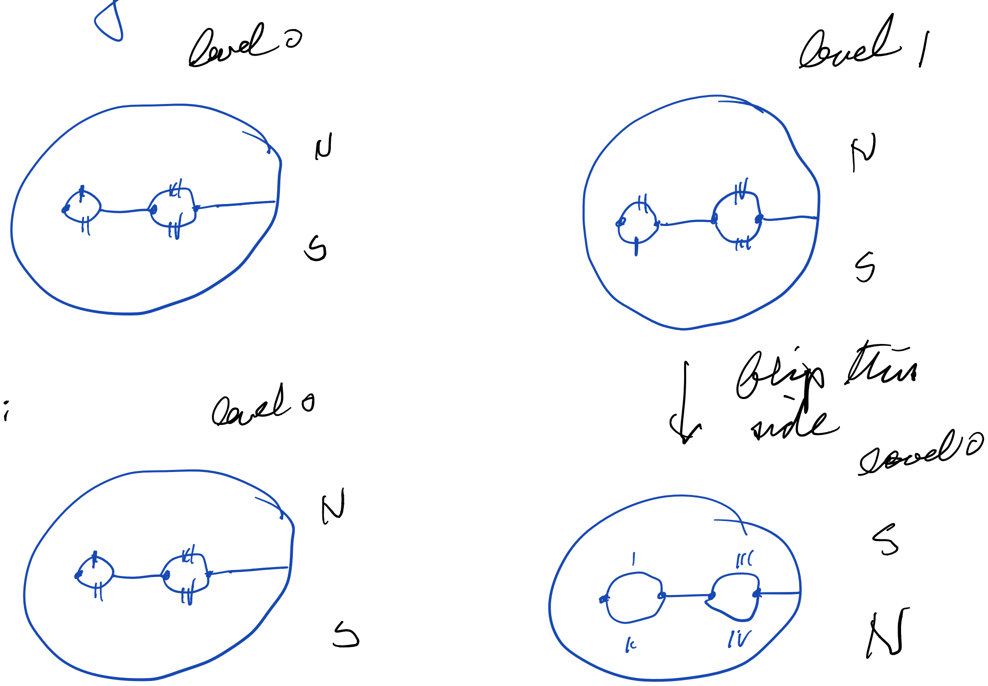
$\psi_a = \psi_e$
$\psi'_w = \psi'_e$

ψ $\quad w_0 h(\beta_j) = w_0 h(\gamma_j) \sim h(\beta_j) = h(\gamma_j) \sim h(\beta_j \cdot \gamma_j^{-1}) = 1$

ψ' $\quad -w_0 h(\beta_j) = -w_0 h(\gamma_j) \sim$

Schematically:

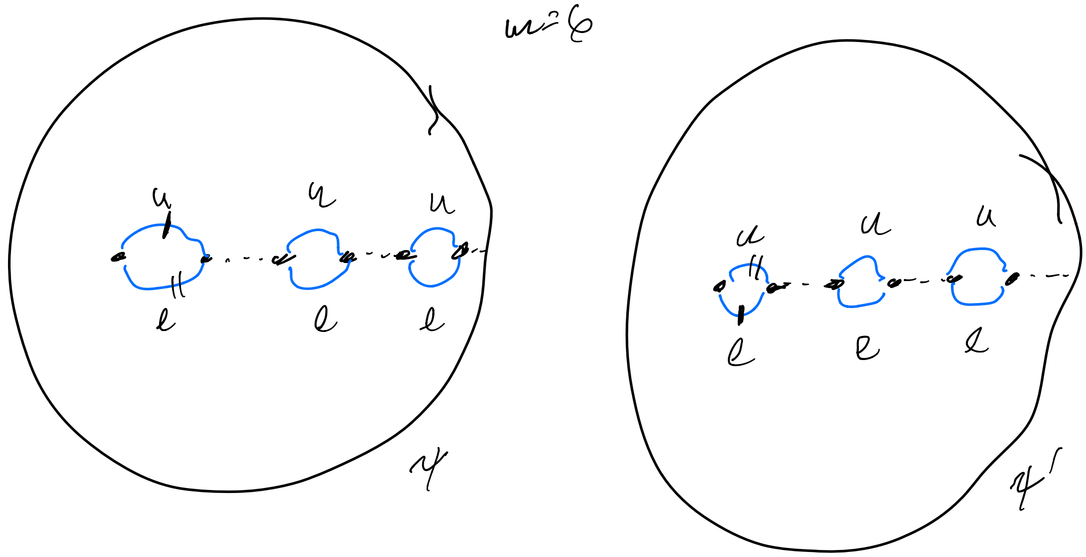
Redraw the picture:



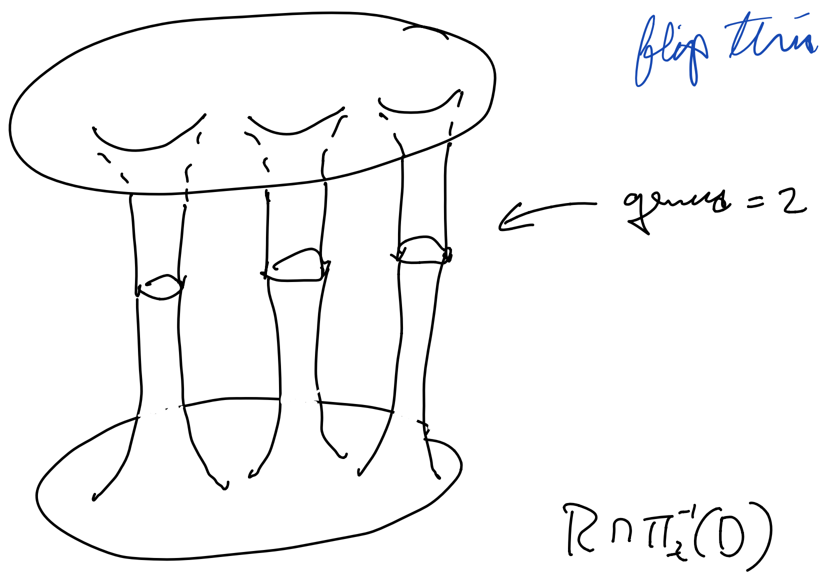
(Like a parking garage with two levels and a north half and a south half. Sometimes going from north to south keeps you on same level

Case 1 m is even

$m=6$



flip this copy



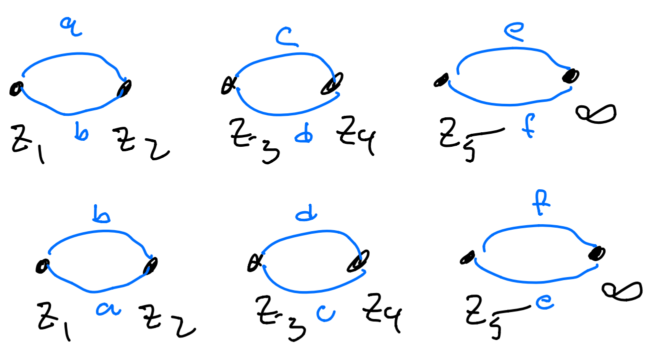
$R \cap \pi_2^{-1}(D)$

Our surface is homeomorphic to a surface of genus $\frac{m}{2}$ with 2 boundary components.

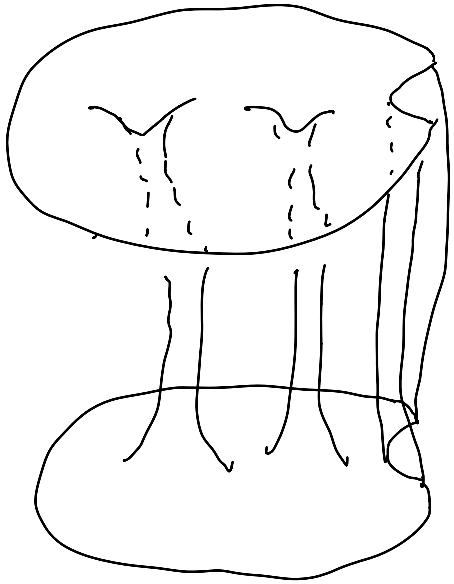
R is homeomorphic to a closed surface of genus $\frac{m}{2}-1$ with 2 pts. removed.

If m is odd:

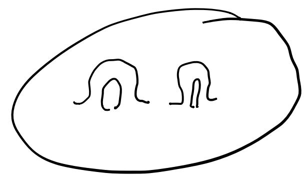
Think of our base surface as $\mathbb{C}_\infty$ instead of $\mathbb{C}$.

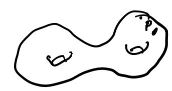


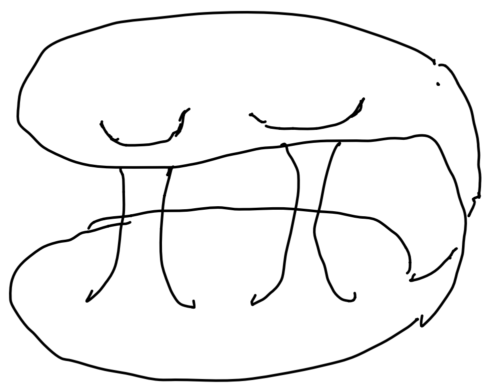
R is homeomorphic to a surface of genus $\frac{m-1}{2}$ with one point removed.



genus = 2.







$\deg(P) = 1.$ V is a copy of $\mathbb{C}$ $g=0$

$\deg(P) = 2.$ V is a cylinder $g=0$

$\deg(P) = 3$ V is a torus minus 1 pt. $g=1$

$\deg(P) = 4$ V is a torus minus 2 pts. $g=1$

Theorem.

$\deg(P) = 2n+1$ V is a surface of genus n minus 1 pt.

$2n$ surface of genus $n-1$ minus 2 pts.