Cor. If $f: R \to S$ is hol. and non constant then $g(S) \le g(R)$. If $g(R) \ge 2$ and $g(R) = g(S)$ then f is a conformal equivalence. If $g(R) = 1$ and $g(S) = 1$ then f is a covering map.

Example sheet

Proof.

Recall

$$\chi(R) = d \cdot \chi(S) - \underbrace{\sum_{\substack{p \in R \\ V_f(p) > 1}} (V_f(p) - 1)}_{\le 0}.$$

≥ 0

May assume $\chi(S) \le 0$ otherwise $\chi(S) = 2$, $g(S) = 0$ and the conclusion is automatically true.

We have $\chi(R) \le d \cdot \chi(S)$. Since $\chi(S) \le 0$

$d \cdot \chi(S) \le \chi(S)$ so $\chi(R) \le \chi(S)$.

This implies that

$$-\chi(R) \ge -\chi(S)$$

$$1 - \frac{\chi(R)}{2} \ge 1 - \frac{\chi(S)}{2}$$

$$g(R) \ge g(S).$$

$g = 1 - \frac{\chi}{2}$

$\frac{\chi}{2} + g = 1$

$\frac{\chi}{2} = 1 - g$

$\chi = 2 - 2g$

Next topic for 2020

At this point we have at least two different ways to produce families of Riemann surfaces of genus 1 (homeomorphic to the torus).

① Elliptic curves $= \{(z,w) : w^2 = P(z)\}$ and $\deg P = 3$ or 4. Distinct roots.

② $\mathbb{C}/\Lambda$, $\Lambda = \{m\lambda_1 + n\lambda_2 : m, n \in \mathbb{Z}\}$ $\lambda_1, \lambda_2 \in \mathbb{C}$

λ_1, λ_2 linearly independent over $\mathbb{R}$.

general question

Can we relate these two constructions?

Are they producing families of holomorphically (or conformally) equivalent curves? In order to address this we want to focus on conformal (holomorphic) invariants of our surface.

An important invariant is the collection (field) of meromorphic functions (and meromorphic 1-forms) on our surface.

We start by studying meromorphic functions on $\mathbb{C}/\Lambda$ where Λ is a lattice in $\mathbb{C}$.

If $g : \mathbb{C}/\Lambda \longrightarrow \mathbb{C}_\infty$ is meromorphic then the composition $f = g \circ \pi$ $\mathbb{C} \xrightarrow{\pi} \mathbb{C}/\Lambda \xrightarrow{g} \mathbb{C}_\infty$ is a meromorphic function on $\mathbb{C}$ which is invariant under the action of Λ.

alg. geom insight: secrets of R are encoded in its meromorphic func.

Recall that that a non-constant meromorphic function on a compact Riemann surface must have at least one pole since it cannot be holomorphic. If it has just one pole then it must be a pole of order at least 2.

A formal way of constructing a doubly periodic function with just one pole of order ℓ on $\mathbb{C}/\Lambda$ is to consider $\mathcal{E}_\ell(z) = \sum_{\lambda \in \Lambda} (z-\lambda)^{-\ell}$ for $\ell \geq 2$.

If we can make sense of this sum then the result should be Λ invariant and have a pole of order ℓ on the points in the lattice Λ.

Step 1 is to restrict our attention to a large disk $D_R = \{|z| < R\}$. $\mathcal{E}_\ell$ will have some poles in D_R and we deal with these separately.

Let $\Lambda_R = \{\lambda \in \Lambda : |\lambda| < R\}$.

Write

$$\sum_{\lambda} (z-\lambda)^{-\ell} = \sum_{\lambda \in \Lambda_R} (z-\lambda)^{-\ell} + \sum_{\lambda \in \Lambda - \Lambda_R} (z-\lambda)^{-\ell}$$

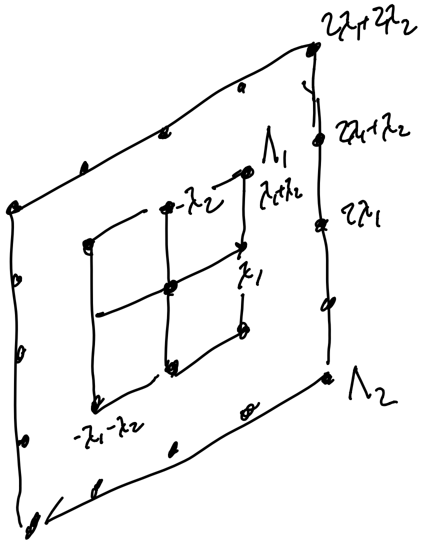
The first term makes sense since it is a finite sum and gives a meromorphic function on D_R.

We want to show that the remaining series converges inside D_R to a holomorphic function. We can try the following:

Theorem. (Weierstrass M-test). Let $W \subset \mathbb{C}$, let $f_n : W \to \mathbb{C}$ be a sequence of holomorphic functions on W. Suppose there is a sequence of positive real numbers M_n such that $|f_n(z)| \leq M_n$ and $\sum M_n < \infty$ then $\sum_n f_n(z)$ converges uniformly to a holomorphic function f and $f'(z) = \sum_n f_n'(z)$.

Let $\Lambda_k = \{ m\lambda_1 + n\lambda_2 : \max\{m, n\} = k \}$

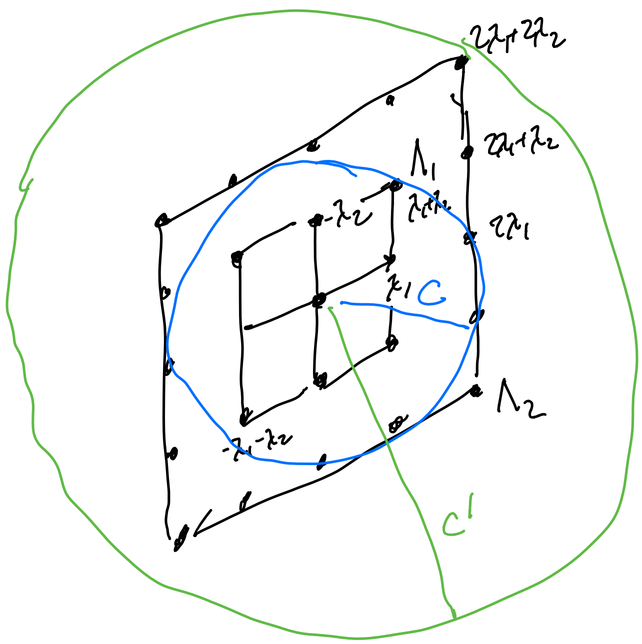
Formally we can write $\sum_{\lambda \in \Lambda} (z-\lambda)^{-l} = \sum_{k \geq 0} \sum_{\lambda \in \Lambda_k} \frac{1}{(z-\lambda)^l}$.



Let P_k be the convex hull of Λ_k.

$$P_k = \{ r\lambda_1 + s\lambda_2 : r, s \in \mathbb{R},\ \max\{r, s\} \le k \}$$

Let c be the radius of the largest disk contained in P_1 and c' be the radius of the largest disk containing P_1.



So

$$D_c \subset P_1 \subset D_{c'}.$$

Since things scale linearly we have:

$$D_{k\cdot c} \subset P_k \subset D_{k\cdot c'}$$

In order to apply the M-test we want to estimate the size of the term $\sum_{\lambda \in \Lambda_k} \frac{1}{(z-\lambda)^\ell}$.

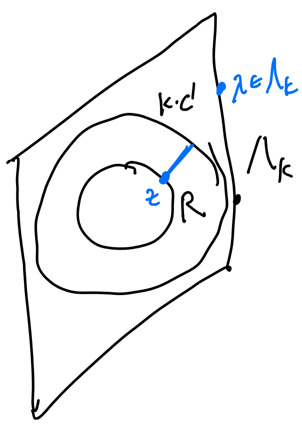
Comes from earlier construction.

Choose k_0 so that $P_{k_0} \supset D_{2R}$. For $k \ge k_0$ and $\lambda \in \Lambda_k$ if $z \in D(0,R)$ then

$d(z,\lambda) \ge R$ so $|z-\lambda| \ge k\cdot c' - R$ $\quad |z-\lambda|^{-\ell} \le (k\cdot c' - R)^{-\ell}$

The # of lattice points in Λ_k is $4(k-1)$.

$$\left|\sum_{\lambda\in\Lambda_k}\frac{1}{(z-\lambda)^{\ell}}\right| \le \frac{4(k-1)}{|k\cdot c'-R|^{\ell}} \le \frac{C_0}{k^{\ell-1}}$$ for some C_0.



This is summable if $\ell-1 \ge 2$ or $\ell \ge 3$.

Prop. For $\ell \ge 3$ the formula for E_ℓ gives a well defined meromorphic function on $\mathbb{C}$ with poles of order 3 at the points of Λ.

Let's check that $\mathcal{E}_\ell$ is translation invariant for $\ell \geq 3$.

Fix $R > 0$. Let $\lambda_0 \in \Lambda$. Want to show that $\mathcal{E}_\ell(z+\lambda_0) = \mathcal{E}_\ell(z)$

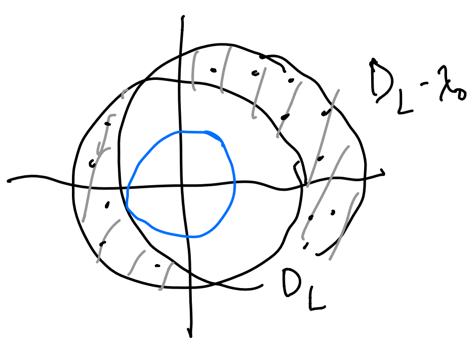
For $z \in D_R$ we can find an $L > 2R$ with

$$\left| \mathcal{E}_\ell(z) - \sum_{\lambda \in D_L \cap \Lambda} (z-\lambda)^{-\ell} \right| \leq \varepsilon \quad \text{and}$$

$$\left| \mathcal{E}_\ell(z+\lambda_0) - \sum_{\lambda \in D_L \cap \Lambda} (z+\lambda_0-\lambda)^{-\ell} \right| = \left| \mathcal{E}_\ell(z+\lambda_0) - \sum_{\lambda \in D_L - \lambda_0} (z-\lambda)^{-\ell} \right| \leq \varepsilon$$

So

$$\left| \mathcal{E}_\ell(z+\lambda_0) - \mathcal{E}_\ell(z) \right| \leq \underbrace{\left| \sum_{\lambda \in D_L \triangle (D_L - \lambda_0) \cap \Lambda} (z-\lambda)^{-\ell} \right|}_{\text{Bounded by the tail of } \sum_{j \geq J} \omega_j} + 2\varepsilon$$



where $J \to \infty$ as $L \to \infty$.

Letting $\varepsilon \to 0$ causes $L \to 0$ and we see that $\mathcal{E}_\ell(z+\lambda_0) = \mathcal{E}_\ell(z)$ for $z \in D_R$. Let $R \to \infty$.

$\mathcal{E}_\ell$ mer. invariant well defined mer. on $\mathbb{C}/\Lambda$.

This completes the construction of $\mathcal{E}_\ell$ $\ell \geq 3$.