We have here established a holomorphic isomorphism between the Riemann surfaces $\mathbb{C}/\Lambda$ and V_Q. Recall that historically V_Q was introduced as a way of dealing with the elliptic integral:

$$\int \frac{dz}{\sqrt{Q(z)}}$$

or the corresponding integral

$$\int \frac{dz}{w}$$

on the surface $w^2 = Q(z)$.

We will now show that the map $u \mapsto (\wp(u), \wp'(u))$ does more. It establishes an isomorphism from the pair $\mathbb{C}/\Lambda$ with the hol. 1-form du and the surface V_Q with the holomorphic 1-form $\frac{dz}{w}$.

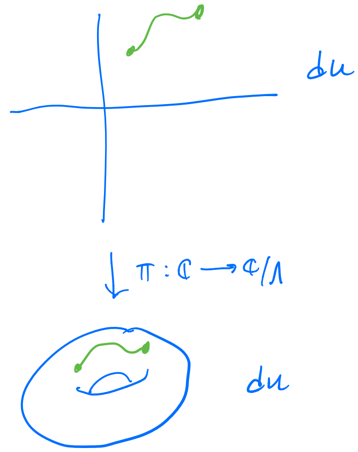
(Note we are writing the variable on $\mathbb{C}$ as u rather than z.)

Prop. $\phi^*\left(\frac{dz}{w}\right) = du$

Proof.
$$\begin{aligned}\phi^*\left(\frac{dz}{w}\right) &= \phi^*(dz)\cdot\frac{1}{\phi^*(w)} \\ &= d\,\phi^*(z)\cdot\frac{1}{\phi^*(w)} \\ &= \frac{d\,P(u)}{P'(u)} = \frac{\frac{d}{du}P(u)du}{P'(u)} \\ &= \frac{P'(u)du}{P'(u)} \\ &= du.\end{aligned}$$

Recall that hol. 1-forms were introduced to give us a theory of integration on Riemann surfaces. Having established a correspondence between 1-forms on Riemann surfaces we also get a correspondence between integration problems.

Integration on $\mathbb{C}/\Lambda$ is relatively straightforward. Let's discuss it.



Say we have a path γ on $\mathbb{C}/\Lambda$. $\gamma : [a,b] \to \mathbb{C}/\Lambda$.

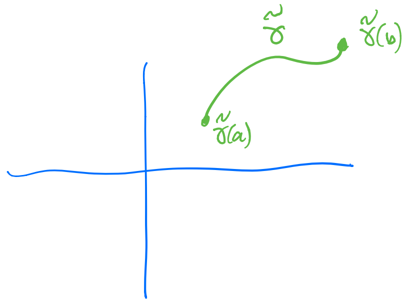
We can lift it to to $\mathbb{C}$: $\tilde{\gamma} : [a,b] \to \mathbb{C}$ so that $\pi \circ \tilde{\gamma} = \gamma$.

This lift is not unique. Two lifts differ by a deck transformation $u \mapsto u + \lambda$ for $\lambda \in \Lambda$.

$$\int_{\gamma} du = \int_{\tilde{\gamma}} du = \int_a^b \tilde{\gamma}'(t)\,dt = \tilde{\gamma}(t)\Big|_{t=a}^{t=b} = \tilde{\gamma}(b) - \tilde{\gamma}(a).$$

(Note that changing the lift gives $(\tilde{\gamma}(b)+\lambda) - (\tilde{\gamma}(a)+\lambda) = \tilde{\gamma}(b) - \tilde{\gamma}(a)$)

What happens when we integrate over a loop in the torus?

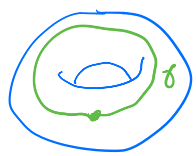


In this case $\tilde\gamma(b) = \tilde\gamma(a)+\lambda$ for some $\lambda\in\Lambda$. So

$$\int_\gamma du = \int_{\tilde\gamma} du = \tilde\gamma(b)-\tilde\gamma(a)$$

$$= \tilde\gamma(a)+\lambda-\tilde\gamma(a)$$

$$= \lambda.$$



So the integral $\oint du$ over any loop gives an element of Λ. The converse is also true: for any element λ of Λ there is a loop γ so that $\int_\gamma du = \lambda$.

Proof. Given λ choose $\tilde\gamma(t) = t\lambda$ for $t\in[0,1]$. Then define $\gamma(t)$ to be $\pi\circ\tilde\gamma(t)$. We get $\int_\gamma du = \int_{\tilde\gamma} du = \tilde\gamma(1)-\tilde\gamma(0) = \lambda-0 = \lambda$.

Now we take the insights gained by looking at integration in the torus and apply them to integration of the form $\frac{dz}{w}$ on V_Q.

Prop. $\Lambda = \{ \int_\gamma \frac{dz}{w} : \gamma \text{ is a loop in } V_Q \}$

Proof. $\Phi : \mathbb{C}/\Lambda \longrightarrow V_Q$ is a holomorphic bijection.

Given a loop α in V_Q then there is a loop $\gamma \in \mathbb{C}/\Lambda$ $\gamma = \Phi^{-1} \circ \alpha$ which maps to it

$$\int_\alpha \frac{dz}{w} = \int_{\Phi(\gamma)} \frac{dz}{w} = \int_\gamma \Phi^*\left(\frac{dz}{w}\right) = \int_\gamma du \in \Lambda$$

Conversely given any $\lambda \in \Lambda$ there is a loop $\gamma_\lambda \in \mathbb{C}/\Lambda$ with $\int_{\gamma_\lambda} du = \lambda$ so letting $\alpha = \Phi \circ \gamma_\lambda$ we get $\int_\alpha \frac{dz}{w} = \int_{\gamma_\lambda} du = \lambda$.

Given a curve V_Q we can determine which lattice it comes from by integrating $\frac{dz}{w}$.
Given Λ we can determine which V_Q it corresponds to by calculating g_2, g_3.

Proposition. The inverse of the holomorphic bijection

$$\Phi : \mathbb{C}/\Lambda \longrightarrow V_Q$$

is given by

$$\Phi^{-1}(p) = \Lambda + \int_{(\infty,\infty)}^{p} \frac{dz}{w}.$$

(Thinking of $\mathbb{C}/\Lambda$ as a space of cosets.)

Proof. Let $p \in V_Q$. Note that we have not specified a path on the right hand side. Say we choose two paths γ_1, γ_2 from (∞,∞) to p then

$$\int_{\gamma_1} \frac{dz}{w} - \int_{\gamma_2} \frac{dz}{w} = \int_{\underbrace{\gamma_1 \cdot \gamma_2^{-1}}_{\text{loop}}} \frac{dz}{w} \in \Lambda \quad \text{so}$$

the result is well defined modulo elements of Λ. Now if we think of $\mathbb{C}/\Lambda$ as being the space of cosets $p+\Lambda$ then the integration map is well defined.

Suppose $\Phi(\Lambda + u_0) = p$ and $\gamma : [a,b] \to V_Q$ is a path from (∞,∞) to p. Then $\alpha = \Phi^{-1}(\gamma)$ is a path in $\mathbb{C}/\Lambda$ from $\Lambda + 0$ to $\Lambda + u_0$. Let $\tilde{\alpha}$ be a lift of α to $\mathbb{C}$ so that $\tilde{\alpha}(a) = 0$. Now $\tilde{\alpha}(b)$ maps to p in $\mathbb{C}/\Lambda$ so it has the form $p + \lambda$ for some $\lambda \in \Lambda$:

$$\begin{aligned}
\int_\gamma \frac{dz}{w} &= \int_{\Phi^{-1}(\gamma)} \Phi^*\left(\frac{dz}{w}\right) \\
&= \int_{\Phi^{-1}(\gamma)} du \\
&= \int_{\tilde{\alpha}} du \\
&= \tilde{\alpha}(b) - \tilde{\alpha}(a) = (p+\lambda) - 0 \in p + \Lambda.
\end{aligned}$$

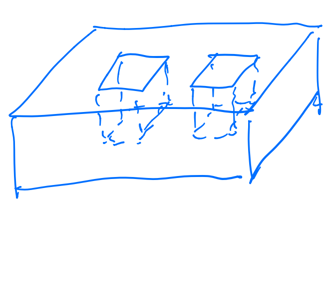
Note that integration on V_Q is pretty concrete.

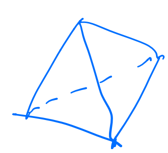
$$\int_\gamma \frac{dz}{\sqrt{Q(z)}}$$

Choose a square root of $w_0 = \sqrt{Q(\gamma(a))}$ then as long as γ does not go through the zeros of Q we can lift this branch of the curve to V_Q and consider

$w_0(t) = \sqrt{Q(\gamma(t))}$. Thus we can integrate $\int \frac{dz}{w_0(t)}$?

Recall our construction of Riemann surface structures on boundaries of polyhedra in $\mathbb{R}^3$.





First step was to construct polygons P_j in $\mathbb{C}$ and (orientation preserving) isometries ψ_j from the P_j to the faces of the polyhedron.

Certain pairs of polyhedra have the property that $\psi_j(P_j)$ and $\psi_k(P_k)$ meet along edges.

We can then make a topological model of our surface by recording identifications of appropriate edges of the $P_j \subset \mathbb{C}$.

If we identify appropriate edges of the P_j we get a topological model $\bigcup_j P_j/\sim$.

We then built an atlas. Charts at non-vertex points compatible with the geometry of $\cup P_j$.

Charts at vertex points have form $z \mapsto z^\alpha$ α related to the cone angle.

Note that our construction is a little more abstract than we admitted.

We can start with polygons in the plane

and glue by isometries of edges to
build a Riemann surface even if the
original polygons do not come from
the boundary of a polyhedron in $\mathbb{R}^3$.

One of these abstract situations arises
in connection with the Weierstrass function $\wp$.