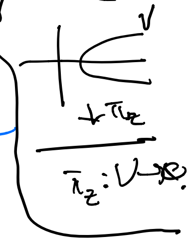
Local behaviour of holomorphic functions between Riemann surfaces.

Examples: Functions from $U \subset \mathbb{C}$ to $\mathbb{C}$. Coordinate projection functions π_z, π_w for V.



Let f be a holomorphic function

$f : U \subset \mathbb{C} \longrightarrow \mathbb{C}$. At $z_0 \in U$ we can write

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

Either all $a_j = 0$ for $j \geq 1$ or there is a smallest j with $a_j \neq 0$ in which case

$$f(z) = a_0 + (z-z_0)^n \left(a_n + a_{n+1}(z-z_0) + \dots\right)$$

$$f(z) = f(z_0) + (z-z_0)^n g(z) \qquad g(z_0) = a_n \neq 0.$$

Let us define $v_f(z_0) = n$ in the second case and $v_f(z_0) = \infty$ in the first case.

We call v_f the valence.

Beardon p. 8

Prop. If f is holomorphic and non-constant on a connected domain U then at each $z \in U$ there is some (finite) j so that

$$f(z) = f(z_0) + (z-z_0)^n \left(a_N + a_{N+1}(z-z_0) + \cdots\right)$$

with $a_N \neq 0$.

(In other words $v_f(z)$ is finite at every point of U or $v_f(z) = \infty$ at every pt. of U and f is constant.)

Proof. Cauchy's integral representation tells us that the coefficients a_j vary continuously with z. Thus the set where $a_j = 0$ is closed.

So the set where all a_j vanish is closed.

On the other hand the set where all a_j vanish is open by analyticity. Since this set is not all of U it is empty by connectivity.

So the set of points with ∞ valence is all of U or empty. If the set is all of U then f is locally constant at each point. In this case say $f(z)=w_0$ for some $z \in U$. The set of z with $f(z)=w_0$ is open and closed and non-empty so all of U.

Corollary. If f is not constant then for each $c \in \mathbb{C}$ the set of z in U with $f(z)=c$ is isolated.

Proof. Say $f(z_0)=c$ then near z_0

$$f(z) = f(z_0) + (z-z_0)^n g(z) \text{ with } g(z_0) \neq 0.$$

so $\overset{f(z)-c}{\overset{\shortparallel}{f(z)-f(z_0)}} = (z-z_0)^n g(z)$. If $z \neq z_0$ is close to z then this difference is non-zero near z_0 by the cont. of g.

(R, S connected, Hausdorff)

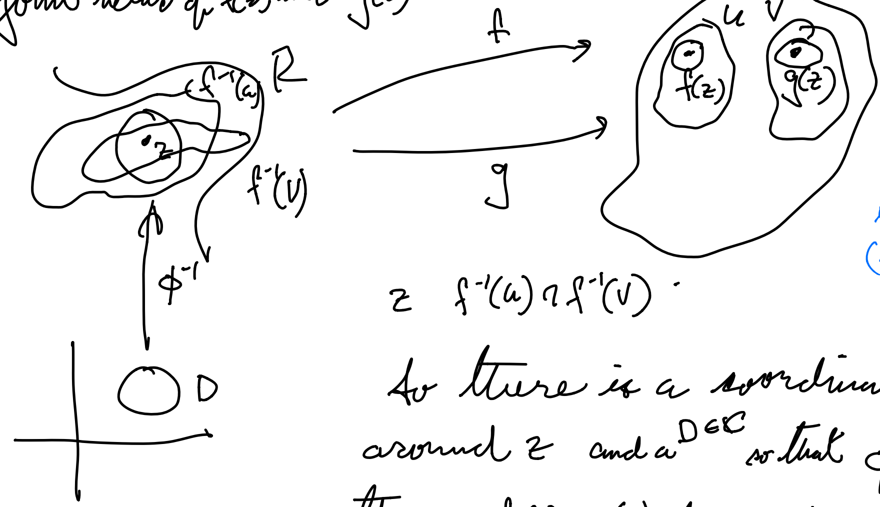
Theorem. Let R and S be Riemann surfaces and suppose that f and g are holomorphic maps of R to S. Then either $f = g$ at every point of R or $f = g$ at isolated points.

special case:

Proof. If $f, g : U \xrightarrow{\subset \mathbb{C}} \mathbb{C}$ consider $f - g = 0$.
Apply previous result.

General case:

Pick $z \in R$. There are two possibilities. If $f(z) \neq g(z)$ then we can find disjoint nbds of $f(z)$ and $g(z)$ in S.

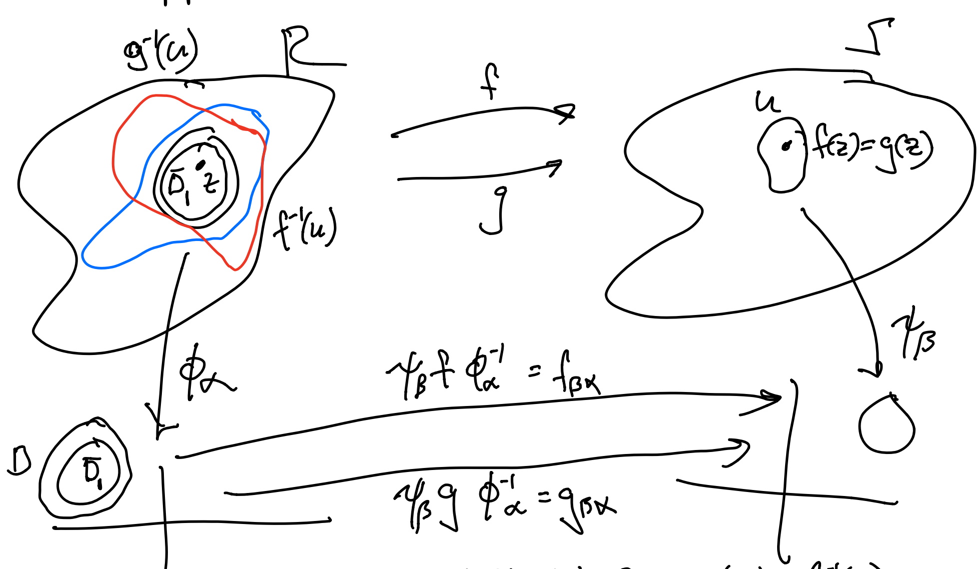


just using topology. (Hausdorff-ness)

$z \quad f^{-1}(u) \cap f^{-1}(V)$.

So there is a coordinate chart around z and a $D \in \mathbb{C}$ so that $\phi(D) \subset f^{-1}(u) \cap f^{-1}(V)$ thus $f(z) \neq g(z)$ for $z \in \phi(D)$.

Now suppose $f(z) = g(z)$. Let u be a nbd. of $f(z) = g(z)$.



There is a compact disk $\bar{D}$ so that $\phi_\alpha^{-1}(\bar{D}) \subset g^{-1}(u) \cap f^{-1}(u)$.

Applying the special case result to $f_{\beta\alpha}$ and $g_{\beta\alpha}$ we see that the set of points where $f=g$ in D is consist of isolated points. So there are only finitely many points where $f(z)=g(z)$ in $\overline{D}$.

Now let A be the set of points p in R which have a nbd N_p where $f=g$ at finitely many points in N_p. A is open.

Let B be the set of points with nbds where $f=g$ throughout N. B is open.

A and B are open and disjoint so $R=A$ or $R=B$ by connectivity.

Cor. If R is a Riemann surface then the collection of meromorphic functions on R (other than $f(R)=\infty$) is a field iff R is connected.

Proof. If f, g are meromorphic functions $p \in R$ and $f(p), g(p) \neq \infty$ we can add multiply functions so the set of functions forms a ring. If g has a discrete set of zeros we can form $\frac{1}{g}$. At a pole or

Recall that if f is not locally constant then $f(z) = a_0 + \dots a_k z^k + a_{k+1} z^{k+1} \dots$ $\quad a_k \neq 0$, $k = \nu_f(0)$.

Lemma. Let f be a holomorphic function on an open nbd. U of 0 in $\mathbb{C}$ with $f(0)=0$ but f not identically 0. Then there is a disk $D \subset U$ centered at 0 and

($\nu_f(0)$ defined and finite)

a holomorphic function $\cancel{g}$ h with $\cancel{g'(0)}$ $h'(0) \neq 0$ $\quad \nu_g(0)=1$

and $f(z) = h^k(z)$ on D where $k = \nu_f(0)$.

Special case of previous discussion.

Proof. $f(z) = a_k z^k + a_{k+1} z^{k+1} + \dots$ $\quad a_k \neq 0$, $a_0 = f(0) = 0$, $a_1 \dots a_{k-1} = 0$.

$$f(z) = a_k z^k \underbrace{(1 + b_1 z + b_2 z^2 + \dots)}_{g(z)} \quad \text{where } b_j = \frac{a_{k+j}}{a_k}.$$

Assume OE $\subset U$ is small enough that

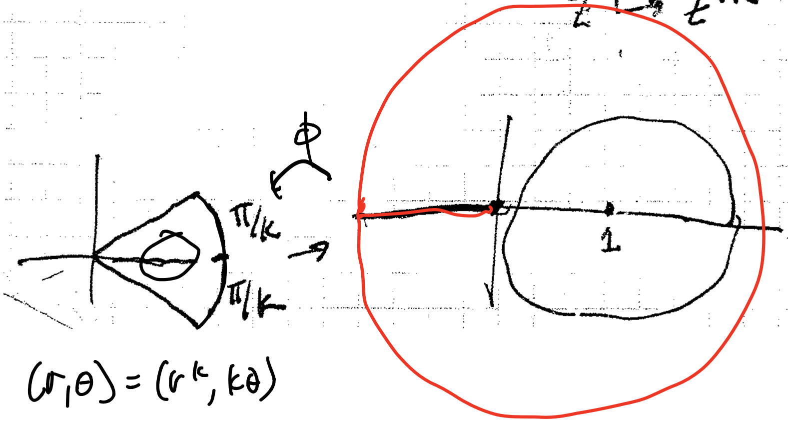
$\left|\sum_{j \geq 1} b_j z^j\right| < 1$ then the image of

$z \mapsto (1 + b_1 z + b_2 z^2 + \dots)$ is contained

in a disk in $\mathbb{C} - \{0\}$. In particular we can choose a branch of $\phi(z)$

$z \mapsto z^{1/k}$ in this disk and $\quad \phi^k(z) = z$.

define



$$h(z) = a_k^{1/k}\, z\, g^{1/k}(z) \qquad \phi \circ g$$

$(r^{1/k}, \theta/k) \leftarrow\!\!\mapsto (r,\theta)$

$\phi((r,\theta)) = (r^{1/k}, \theta/k)$

$\phi \circ g$

Let $h(z) = a_k^{1/k}\, z\, g(z)$ for some choice of k-th root

then $h^k(z) = a_k z^k (1 + b_1 z + b_2 z + \dots) = f(z)$.

Furthermore $h'(0) = a_k^{1/k} \neq 0$.

Theorem. Let f be a hol. $\mathbb{C}$ valued function in a Riemann surface R. Let $p \in R$ then there is a locally invertible function ϕ_1: nbd of p to $\mathbb{C}$ so that $f = \phi_1^k$ ($f(w) = \phi_1^k(w)$.)