Remarks about thm. from last class.

Showed that for 2 hol. fcns on Riemann surfaces $f=g$ everywhere or $f=g$ at an isolated set of points. Uses connectivity of domain.

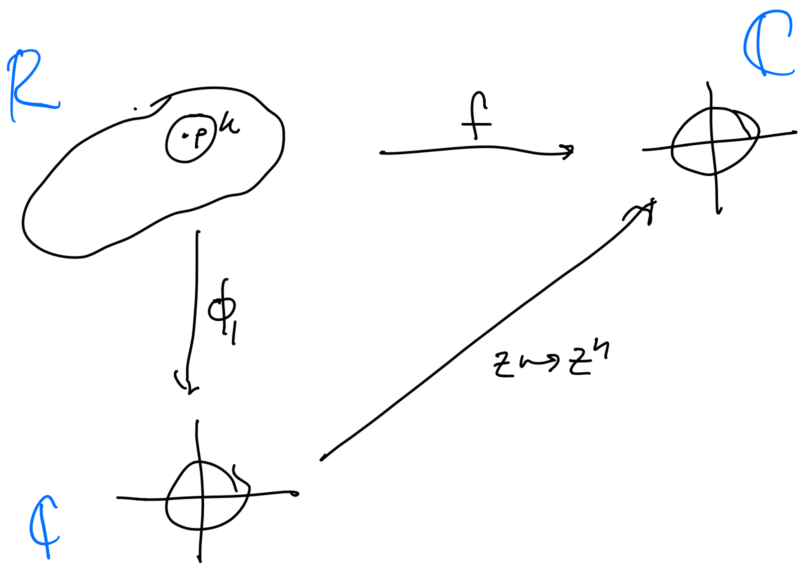
Divide the domain into 2 open sets.

<u>Not</u> the set where $f=g$ and set where $f\neq g$.

(Lemma. If $f: U \to \mathbb{C}$ and $f(0)=0$
Then there is a nbd. of 0 and an h with $h'(0)\neq 0$ so that
$f(z)=h(z)^k$.)

Let's assume Riemann surfaces are connected and Hausdorff.

Theorem. Let f be a non-constant complex valued holomorphic function on a connected Riemann surface R, and assume $f(p)=0$.
Let $p \in R$ then there is an invertible holomorphic function $\phi_1 : \overset{\ni p}{U} \to \overset{\subset \mathbb{C}}{V}$ so that $f(z) = \phi_1^n(z)$ where $U = V_f(p)$.

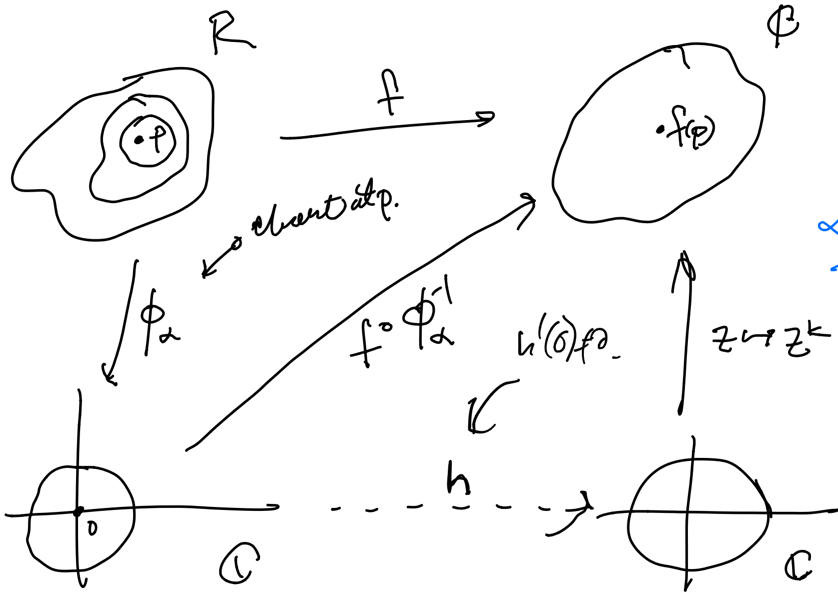


(We can think of ϕ_1 as a chart.)

Proof.

Adjust ϕ_α by adding a const so $\phi_\alpha(p)=0$.
In the atlas?
Could be.

If we add it to the atlas we get an equiv. Riemann surface.



Apply Lemma to get h. (Recall Lemma here.)

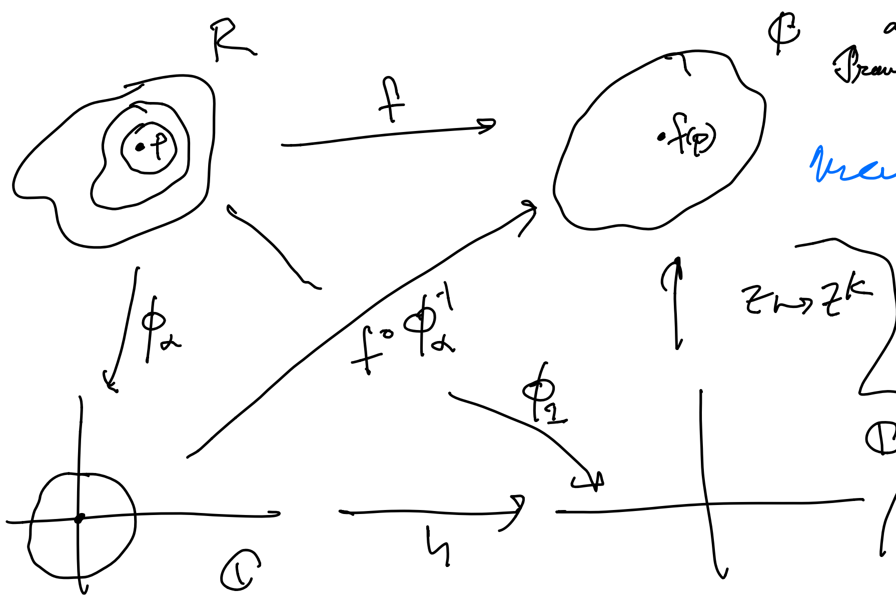
Recall h has valence 1 at 0. Thus h is locally invertible.
$h'(0)\neq 0$

Let $\phi_1 = h\circ\phi_\alpha$.

ϕ_1 is holomorphic and bijective.
ϕ_1 could be in the atlas $\mathcal{A}$.

Throw it in create atlas $\mathcal{A}'$.
Transition fcn. with ϕ_α is hol.

maximal atlases



and is locally invertible by the inverse fcn. thm.

$f=\phi_1^k$ near p.

Theorem. Geometric formulation.

Corresponding atlas of R so that . . .

If $f: R \to S$ is a holomorphic map between Riemann surfaces and $p \in R$ then there are charts ϕ_α with $\phi_\alpha(p) = 0$ and ψ_β with $\psi_\beta(f(p)) = 0$ where

Local model, Choosing best local coord. for a given problem.

$$
\begin{tikzcd}
R \ni p \arrow[r, "f"] \arrow[d, "\phi_\alpha"'] \arrow[dr, "\psi_\beta \circ f"] & S \ni f(p) \arrow[d, "\psi_\beta"] \\
\mathbb{C} \arrow[r, "z \mapsto z^k"'] & \mathbb{C}
\end{tikzcd}
$$

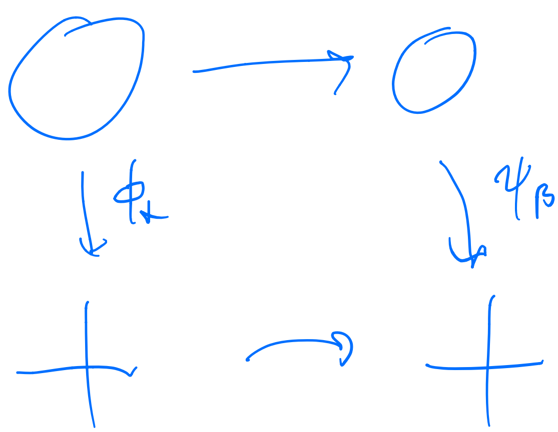
with $k = v(f, p)$.

Proof. There is some chart ψ_1 defined in V_1 with $f(p) \in V_1$. Let $\psi_\beta(q) = \psi_1(q) - \psi_1(f(p))$.

Now consider $\psi_\beta \circ f$ and find a chart ϕ_α in which this function can be written as $z \mapsto z^k$

Corollary. Let $f: R \to S$ be a holomorphic function, not constant and $p \in R$. Then for q suff. close to $f(p)$ the equation $f(z) = q$ has n solutions where n is the valence of f at p when expressed in a chart.

Let $f: R \to S$ be holomorphic. Define $v_f(p)$ to be $v_{f_{\alpha\beta}}(q)$ where $q = \phi_\alpha(p)$.



Corollary. The valence $v_f(z)$ is well defined on manifolds.

$$\boxed{v_{f\circ g}(p) = v_f(g(p)) \cdot v_g(p)}$$

Why is there a question?

Originally defined in terms of power series expansion and the power series expansion depends on the chart.

Equivalent to a geometric property that can be defined without reference to the chart.

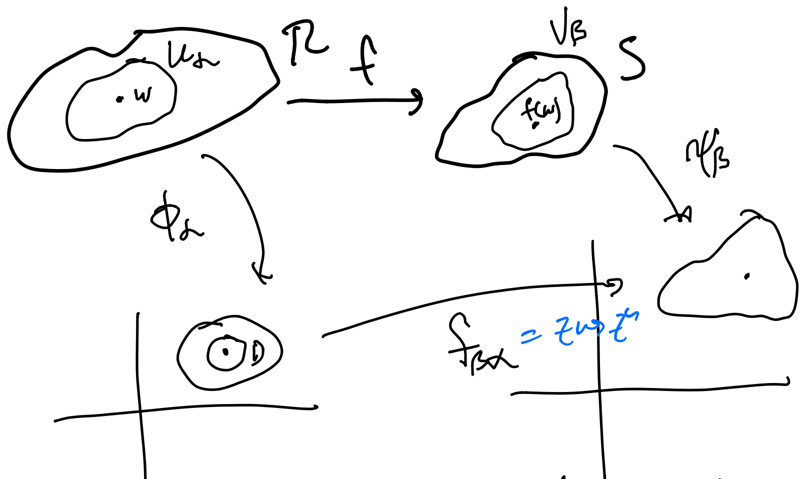
Map $z \mapsto z^n$ takes a nbd. of 0 to <u>a nbd.</u> of 0. Open property.

Theorem. Let R, S (connected) be Riemann surfaces and suppose that $f: R \to S$ is holomorphic but not constant. If $A \subset R$ is open then $f(A)$ is open.

Proof. Say $A \subset R$ is open. Assume A is not empty. Let $w \in A$. We have a coordinate chart φ mapping a nbd. of w, U to $\mathbb{C}$.

Let D be a disk around $\varphi(w)$.

φ^{-1}



Choose a disk D around $\phi_\alpha(w)$ so that $\phi_\alpha^{-1}(D) \subset f^{-1}(V_\beta)$.

By the previous thm. since f is not constant it is not locally constant so $f_{\beta\alpha}$ is not constant. Thus $f_{\beta\alpha}(D)$ is open by the open mapping thm for holomorphic functions from $\mathbb{C}$ to $\mathbb{C}$. So $\varphi_\beta^{-1}(f_{\beta\alpha}(D))$ is open since φ_β is continuous.

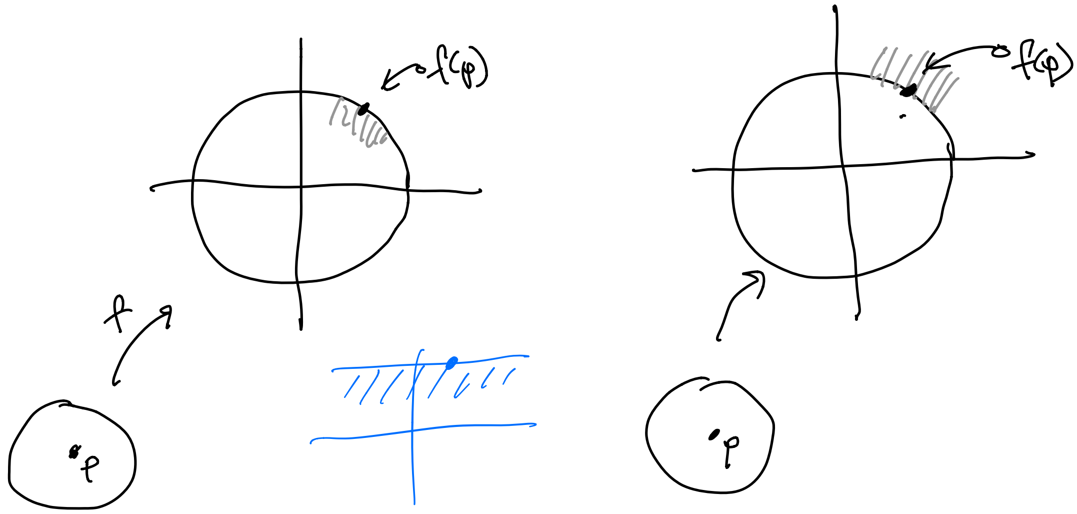
It follows that $f(A)$ contains a nbd. of each of its points.

Theorem. Let $f: R \to \mathbb{C}$ be holomorphic but not constant on a Riemann surface R.

Then $|f|$ has no local maximum.

$|\mathrm{Re}(f)|$ has no local maximum.

This follows from the open mapping theorem.



R, S connected

Theorem. If $f: R \to S$ is analytic but not constant and if R is compact then $f(R) = S$ and so S is compact.

In particular a holomorphic function on a compact surface is constant.