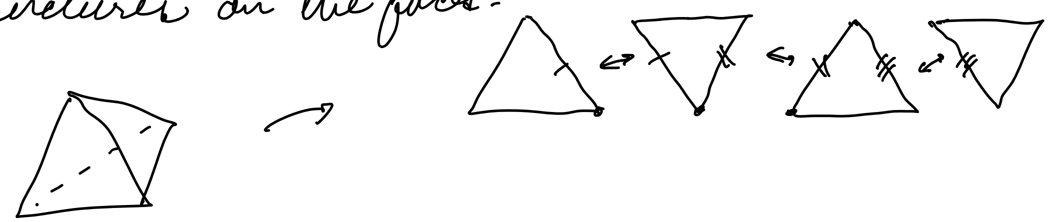
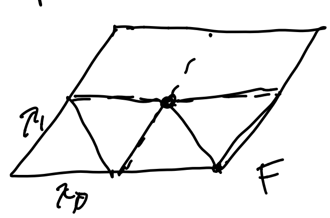
These regions have well defined metrics on their faces. These metrics together with the choice of an orientation give conformal structures on the faces.



We can also think of $\mathbb{C}/\Lambda$ as having not just a conformal structure but a metric structure. Γ action preserves this metric structure.

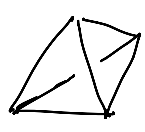
Example.



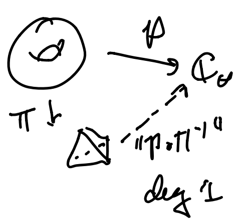
Take $\lambda_0 = 1$, $\lambda_1 = e^{2\pi i/6}$

Consider $\gamma(z) = -z + \lambda_0 + \lambda_1 = -z + 1 + e^{\pi i/3}$.

Fixed pt. is $\frac{1+e^{\pi i/3}}{2}$. $\gamma(F) = F$.

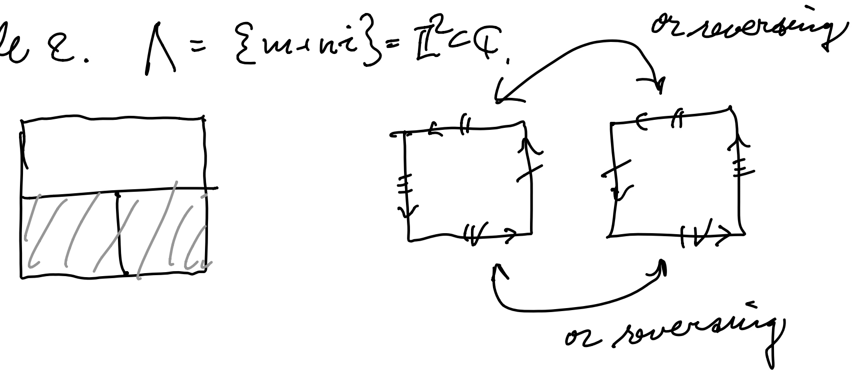


Quotient surface $\mathbb{C}/\Gamma$ can be identified with the boundary of the tetrahedron. We constructed a Riemann surface structure on this surface earlier in the course.



Conformal isomorphism

Example 2. $\Lambda = \{m+ni\} = \mathbb{Z}^2 \subset \mathbb{C}$.



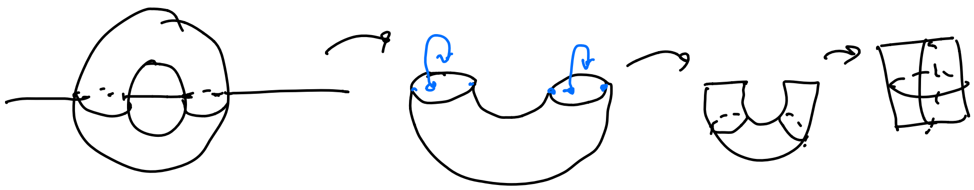
Viewed as a geometric object this is the pillowcase .

Note that it is no longer the boundary of a polygonal region in $\mathbb{R}^3$ but it can still be realized as obtained from polygonal regions in $\mathbb{R}^2$ glued together along their boundaries by isometries.

Note that the images of the "cone points" of $p\circ\pi^{-1}$ are the images of the branch points of p.

$\wp$ induces a conformal

isomorphism from the boundary of the tetrahedron to $\mathbb{C}_\infty$. So this  is conf. isomorphic to $\mathbb{C}_\infty$.



This picture suggests that the points of valence 2 for $\wp$ are the half-lattice points. Let's check this.

$\wp' = -2\mathcal{E}_3$ and $\mathcal{E}_3 = \sum_{\lambda\in\Lambda} \frac{1}{(z-\lambda)^3}$ is an odd function.

If $\lambda_0 \in \Lambda/2$ then

$$\wp'(\lambda_0) = -\wp'(-\lambda_0) = -\wp'(-\lambda_0 + 2\lambda_0) = -\wp'(\lambda_0).$$

So $\wp'(\lambda_0) = 0$. This implies that for $\lambda_0/2$, $\lambda_1/2$ and $\lambda_0+\lambda_1/2$ $\wp$ has valence at least 2. But $\wp$ has degree 2 so the valence must be exactly 2.

Since $\wp$ has a pole of order 2 at 0, $\wp$ also has valence 2 at 0.

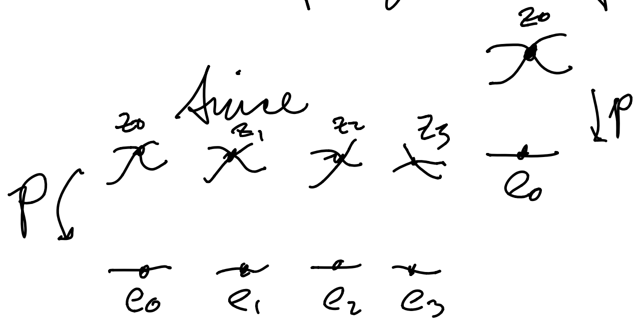
Note that this is consistent with the Riemann-Hurwitz formula:

$$\underset{\substack{\parallel \\ -4}}{\chi(T)} - 2\chi(S^2) = \sum_{V_p(p) \geq 1} 1 - V_p(p) = 4(-1).$$

Prop. The values of $\wp$ at $0, \frac{z_0}{2}, \frac{z_1}{2}, \frac{z_0+z_1}{2}$ are distinct.

Proof. $\wp : \mathbb{C}/\Lambda \to \mathbb{C}_\infty$ has degree 2 and at each half-lattice point it has valence 2, The degree formula gives $2 = \sum_{p : \wp(p) = q} V_p(\wp) = 2 + \sum_{p' \neq p} V_{p'}(\wp)$

Since



Any additional inverse give additional positive contributions to the degree.

Prop. $(\wp'(z))^2 = 4\wp^3(z) - g_2 \wp(z) - g_3$

for certain constants g_2 and g_3 that depend on the lattice Λ.

Proof. $\wp(z) - \frac{1}{z^2} = \sum_{\lambda \in \Lambda - \{0\}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}$

vanishes at 0 since each term vanishes at 0 and it is an even function (since $\wp$ and $\frac{1}{z^2}$ are even functions).

Thus $\wp(z) = z^{-2} + \lambda z^2 + \mu z^4 + O(z^6)$

$$\wp'(z) = -2z^{-3} + 2\lambda z + 4\mu z^3 + O(z^5)$$

$$(\wp'(z))^2 = 4z^{-6} - 8\lambda z^{-2} - 16\mu + O(z^4)$$

$$\wp^3(z) = z^{-6} + 3\lambda z^{-2} + 3\mu + O(z^4)$$

$$(\wp'(z))^2 - 4\wp^3(z) = \underbrace{-20\lambda z^{-2} - 28\mu}_{-20\lambda \wp(z) - 28\mu + O(z^3)} + O(z^5)$$

Thus $(\wp'(z))^2 - 4\wp^3(z) + 20\lambda\, \wp(z) + 28\mu$ has no pole at 0 and has value 0 at 0.

Since the only poles of $\wp$ and $\wp'$ occur at lattice points this function has no other poles.

Thus we have a hol. fcn on a compact Riemann surface so it is constant.

Evaluating at 0 we see that it vanishes.

Now set $g_2 = -20\lambda$ and $g_3 = -28\mu$.

Cor. The polynomial $4z^3 - 20\lambda z - 28\mu$ has distinct 0's.