Counting solutions "with multiplicity". $\nu_f(p)$

Say $f(p)=q$ then $\nu_f(p)$ is the # of solutions of $f(p)=q$ "counted with multiplicity".

Remark about the ring of functions. Geometry translates into alg. properties of the ring of functions on a variety into geometric facts about the variety.

Prop. If R is connected then the ring of hol. fcns. on R has no zero divisors.

Proof. Say $fg=0$ but f,g are not 0.

If neither f nor g is zero everywhere then each is zero on a discrete set so $f \cdot g$ is zero on a discrete set and $fg \neq 0$.

(Not true for the ring of cont. fcns. Not true for disconnected surfaces.)

Theorem. Let $f: R \to \mathbb{C}$ be holomorphic but not constant on a Riemann surface R.

Then $|f|$ has no local maximum.

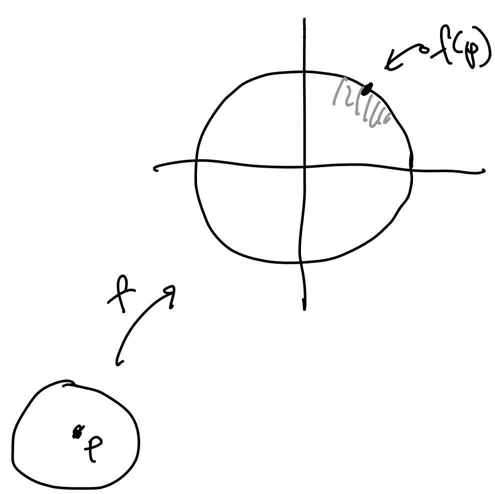
$|\mathrm{Re}(f)|$ has no local maximum.

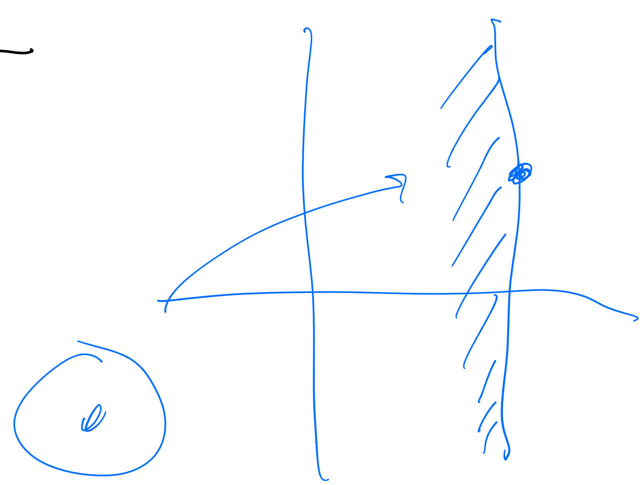
Proof.

This follows from the open mapping theorem.

Say p is a point at which $|f(p)|$ is a maximum.

If f is not constant we could find a point in a nbd. of p with a larger value of $|f(p)|$.





Theorem. If $f: R \to S$ is analytic (R, S connected) but not constant and if R is compact then $f(R) = S$ and so S is compact.
In particular a holomorphic function on a compact surface is constant.

Proof. Image is open, (by open mapping thm) closed by compactness and non-empty, so is all of S so S is compact. If S is not compact f is constant.

(Our idea of looking at the ring of holomorphic functions on a compact surface is not so good.)

Recall:

If f is holomorphic in some domain

$$D = \{z : 0 < |z-w| < r\}$$

then we say f has an isolated singularity at w. D is called a punctured disk.

We can expand f in a Laurent series around w:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-w)^n.$$

Let $N = \inf\{n : a_n \neq 0\}$ then f has a <u>removable singularity</u> if $N \geq 0$, a <u>pole</u> if $-\infty < N < 0$ and an <u>essential singularity</u> if $N = -\infty$.

If f has a <u>removable singularity</u> then we can extend f to a holomorphic function defined at w given by $\sum_{n\geq 0} a_n (z-w)^n$.

If f has a pole then set $m = n - N$ $\quad n = m + N$

$$f(z) = \sum_{n=N}^{\infty} a_n (z-w)^n = \sum_{m=0}^{\infty} a_{m+N} (z-w)^{m+N}$$

$$= (z-w)^N \underbrace{\sum_{m=0}^{\infty} a_{m+N} (z-w)^m}_{\text{Call this } g.}$$

g is holomorphic and $g(w) = a_N \neq 0$.

So $f(z) = \dfrac{g(z)}{(z-w)^n}$ $\quad n \geq 1$ $\quad g(w) \neq 0$.

It follows that $\lim_{z \to w} |f(z)| = \infty$.

If f has an essential sing. then the collection of limiting values is dense in $\mathbb{C}$ by Casoratti-Weierstrass.

P. 10
Beardon

We say a function is meromorphic in a domain $U \subset \mathbb{C}$ if f is holomorphic in $U - \{z_j\}$, each z_j is isolated in U and f has at worst a pole at each z_j.

Meromorphic functions.

Local behavior of a hol. fcn. at an isolated sing. is a top. invariant.

Makes sense to define this for Riem. surfaces.

Since being a pole is a topological property makes sense independently of the chart.

That is to say if $f: R \to \mathbb{C}$ has a pole at $p \in R$ with respect to one chart then it has one with respect to any chart whose domain contains p.

$\phi_{\alpha\beta}$ is a hol. diffeo

Do this in the context of Riemann surfaces. (but for convenience works in a single chart.)

If f is meromorphic in U then we can define an extension of $f^+ : U \to \mathbb{C}_\infty$ by setting $f^+(z) = f(z)$ when z not a pole of f and $f^+(z) = \infty$ when z is a pole of f.

Clear from the prev. discussion that f^+ is continuous.

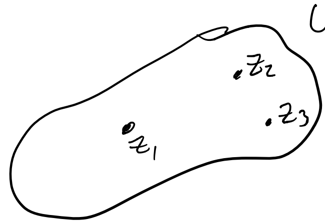
Remark. $U \subset \mathbb{C}$ can be viewed as a Riemann surface where its atlas consists of the single chart $\iota : U \to \mathbb{C}$ corresponding to the inclusion.

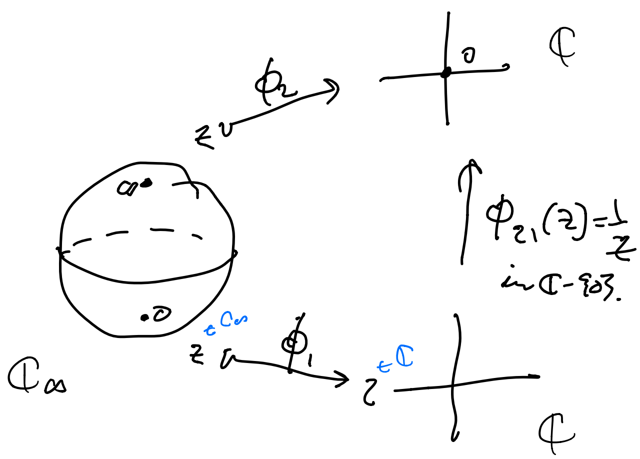
Prop. Viewed as a map between Riemann surfaces $f^+ : R \to \mathbb{C}_\infty$ is a holomorphic map.

Proof, Need to show that the map is holomorphic when expressed in charts.

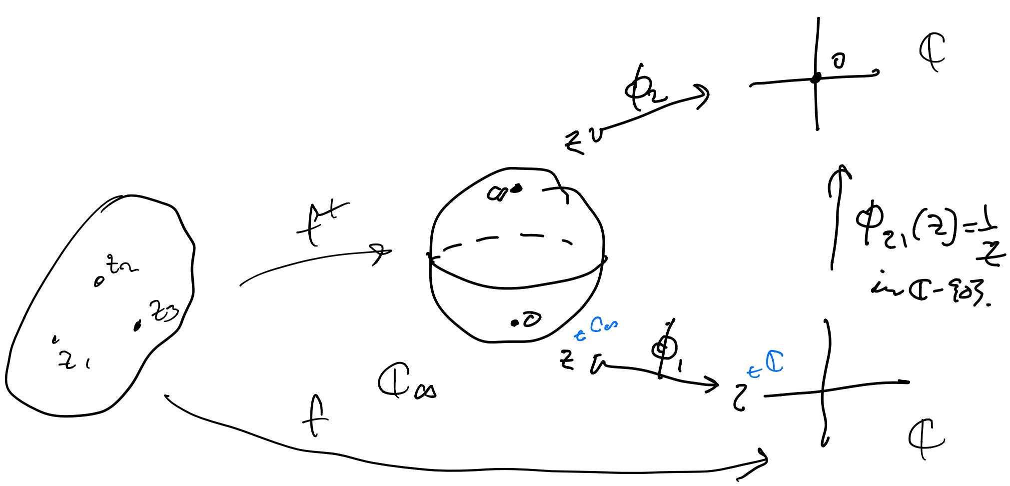
For simplicity we replace R by the range of one of its charts.

$U \subset \mathbb{C}$ Let z_j be poles in U.





We use the chart ϕ_1 to identify $\mathbb{C}_\infty - \{\infty\}$ with $\mathbb{C}$,



We have $\phi_1 \circ f^+(z) = f(z)$ for $z \neq z_j$.

By construction $\phi_1 \circ f^+$ is holomorphic away from the poles.

Consider z in a nbd. of z_j. Want to show that $\phi_2 \circ f^+(z)$ is holomorphic.

For $z = z_j$ $\phi_2(f^+(z)) = \phi_2(\infty) = 0$.

For $z \neq z_j$ $\phi_2(f^+(z)) = \phi_{21} \circ \phi_1 f^+(z)$

$$= \phi_{21}(f(z))$$

$$= \frac{1}{f(z)}$$

Now $f(z) = \dfrac{g(z)}{(z-z_j)^N}$

$g(z_j) \neq 0$

$$= \begin{cases} \dfrac{(z-z_j)^N}{g(z)} & (z \neq z_j) \\ 0 & (z = z_j) \end{cases}$$

This function is holomorphic.

Cor. If R is a Riemann surface then the collection of meromorphic functions on R (other than $f(R)=\infty$) is a field iff R is connected.

Proof. If f, g are meromorphic functions $p \in R$ and $f(p), g(p) \neq \infty$ we can add multiply functions so the set of functions forms a ring. If $g \neq 0$ then g has a discrete set of zeros we can form $\frac{1}{g}$ as a meromorphic function with poles at the zeros of g.

Need to check that we have enough non-infinite values to determine the function.

Only fails when $f \equiv \infty$.

Corollary.

If f is meromorphic on a compact surface then $f(R) = \mathbb{C}_\infty$ unless f is constant.

For example a non-constant meromorphic function always has a zero. (Generalizes the fact that a non-const. poly. in $\mathbb{C}$ always has a root.)

In the case $R = \mathbb{C}_\infty$ we can calculate the field of meromorphic functions.