Corollary. A non-constant meromorphic function on a compact Riemann surface takes on every value.

Proof. A meromorphic function is a holomorphic function $f: R \to \mathbb{C}_\infty$. If it is non-constant it is surjective.

Prop. If R is a Riemann surface then the collection of meromorphic functions on R (other than $f(R)=\infty$) is a field iff R is connected.

Proof. Want to define addition, mult., division of functions by adding, mult., dividing their values $\frac{f}{g}(z) = \frac{f(z)}{g(z)}$. This fails when the values are ∞ or when we are dividing by 0. If f, g are meromorphic but not const $=0,\infty$ then at any z_0 $f(z)=(z-z_0)^n \cdot f_1(z)$ $g(z)=(z-z_0)^m \cdot g_1(z)$

so $\frac{f(z)}{g(z)} = (z-z_0)^{n-m}\frac{f_1(z)}{g(z)}$ has at most an isolated pole.

In the case $R = \mathbb{C}_\infty$ we can <u>calculate</u> the field of meromorphic functions.

How do you build field extension of a field F?

Typically you add an element. If that element satisfies a polynomial with coef. in F then you have an <u>algebraic</u> extension.

(This is where Galois theory works.)

If that element satisfies no polynomial you have a transcendental extension. Typical example $F(x)$, the field of rational functions over $F(x) = \left\{ \frac{a_0 + a_1 x + \dots a_n x^n}{b_0 + b_1 x + \dots b_m x^m} \text{ where denom} \neq 0 \right\}$.

Purely algebraic expression.

Theorem. A function f is meromorphic on $\mathbb{C}_\infty$ if and only if it is a rational function.
(We get $\mathbb{C}(X)$.)

[Notation: If F is a field $F[X]$ is the ring of power series and $F(X)$ is the field of rational functions in X. Completely formal.

Note: Meromorphicity is a local assumption.
Rationality is a global conclusion.
"Locally nice + compactness $\Rightarrow$ algebraic"

Proof. Let f be a rational function.
$f = \frac{P}{Q}$ with $Q \neq 0$ then arguing as above we see that f is meromorphic in $\mathbb{C}$.

To check that f is meromorphic at ∞.

Any $f(\infty)$ is finite.

$$\begin{tikzcd}
\mathbb{C}_\infty \arrow[r, "f"] & \mathbb{C}_\infty \arrow[d, "\varphi_1"] \\
\mathbb{C} \arrow[u, "z \mapsto \frac{1}{z}\ \varphi_2^{-1}"] \arrow[r, "F"] & \mathbb{C}
\end{tikzcd}$$

$$a_0 + \frac{a_1}{z} + \dots \frac{a_n}{z^n}$$
$$\parallel$$

Let $F(z) = f(\frac{1}{z})$. Now $f(1/z) = \dfrac{P(1/z)}{Q(1/z)} = \dfrac{z^m \cdot P(\frac{1}{z})}{z^m \cdot Q(\frac{1}{z})}$.

For M suff. large numerator and denominator are polynomials so F is rational hence meromorphic. | If $f(\infty)=\infty$ we $\div z$ on RHS and look at $\frac{1}{f(1/z)} = \frac{Q(1/z)}{P(1/z)}$. Still rational.

Now assume f is meromorphic. Set of poles is isolated hence finite.
Assume f tends as to a finite value (?) Enough to show for $\frac{1}{f}$ is rational.
Assume f finite at ∞.

If $z_j \in \mathbb{C}$ is a "finite pole" we can write

$$f(z) = \sum_{n=-N}^{\infty} a_n (z-z_j)^n \text{ for } z \text{ near } z_j.$$

The function $f_j(z) = \sum_{n=-N}^{-1} a_n (z-z_j)^n$ is rational and tends to 0 as $z \to \infty$ since it only contains negative powers of $(z-z_j)$.

Let $R(z) = \sum_j f_j(z)$.

Now $f - R$ is holomorphic at each z_j since we have subtracted off the negative powers of $(z-z_j)$ so it is holomorphic in $\mathbb{C}$.

Furthermore $f-R$ is continuous and finite valued at ∞ so it is bounded.

By Liouville theorem $f-R$ is constant so

$f(z) = R(z) + c = \sum f_j(z) + c$

showing that f is rational.

Can we get more examples of compact Riem. surfaces?

We have defined hyper-elliptic surfaces as non-compact Riemann surfaces. We showed that they are top. equiv. to compact surfaces with 1 or 2 pts missing. Do they in fact correspond to Riemann surfaces with missing points?

Step 1 is to compactify them using meromorphic functions.

$$V = \{(z,w) : w^2 = P(z)\}$$

$V \subset \mathbb{C} \times \mathbb{C}$. We can compactify each factor.

$\mathbb{C} \hookrightarrow \mathbb{C}_\infty$.

$$V \subset \mathbb{C} \times \mathbb{C} \subset \mathbb{C}_\infty \times \mathbb{C}_\infty.$$

Write $\bar{V}$ for the closure of V in $\mathbb{C}_\infty \times \mathbb{C}_\infty$.

Claim. $\bar{V} = V \cup$ one point.

Proof. Assume P is not constant.

Say (z_j, w_j) converges to a point in $\bar{V}$. If $z_j \to z_0 \in \mathbb{C}$ then $w_j \to w_0$ with $(z_0, w_0) \in V$. Say that this doesn't happen so $|z_j| \to \infty$. Now $|w_j| = \sqrt{|P(z_j)|} \to \infty$ and $(z_j, w_j) \to (\infty, \infty)$.

$\mathbb{C}_\infty \times \mathbb{C}_\infty$ is a 2 complex dimensional manifold and we have not discussed these. We proceed despite this. We have charts ϕ_1, ϕ_2 for $\mathbb{C}_\infty$ so we can get charts (ϕ_1, ϕ_1) (ϕ_1, ϕ_2) (ϕ_2, ϕ_1) (ϕ_2, ϕ_2) for $\mathbb{C}_\infty \times \mathbb{C}_\infty$. The relevant chart for the point (∞, ∞) is (ϕ_2, ϕ_2). Introduce variables

$u = \frac{1}{z}$ and $v = \frac{1}{w}$.

$$\begin{array}{ccc} \mathbb{C} \times \mathbb{C} & \xrightarrow{(\phi_2, \phi_2)} & \mathbb{C}_\infty \times \mathbb{C}_\infty \\ (u, v) & \longmapsto & \left(\frac{1}{z}, \frac{1}{w}\right) \end{array} \qquad \begin{array}{ccc} \mathbb{C} \times \mathbb{C} & \xrightarrow{(\phi_1, \phi_1)} & \mathbb{C}_\infty \times \mathbb{C}_\infty \\ (z, w) & \longmapsto & (z, w) \end{array}$$