We showed last time that.

$$(P'(z))^2 = 4P^3(z) - 20\lambda P(z) - 28\mu \qquad 4z^3 - 20\lambda z - 28\mu$$

Now set $g_2 = -20\lambda$ and $g_3 = -28\mu$ so

$$(P'(z))^2 = 4P^3 + g_2 P + g_3.$$

Cor. The functions $u \overset{\Phi}{\mapsto} (P(u), P'(u))$, $\mathbb{C}/\Lambda - \{0\}$, from $\mathbb{C}/\Lambda$ to $\mathbb{C}^2$ take values in the curve

$$R = \{(z,w) : w^2 = 4z^3 + g_2 z + g_3.\}$$

Remark. The condition that the zeros of $4z^3 - g_2 z - g_3$ be distinct was required so that $R \subset \mathbb{C}$ be have no singular points and yield a Riemann surface.

Recall that $\bar{R} \subset \mathbb{C}_\infty \times \mathbb{C}_\infty$ is $R \cup \{(\infty,\infty)\}$ and we have constructed a Riemann surface structure for $\bar{R}$, $\bar{R}$.

Prop.

This parametrization Φ extends to a conformal isomorphism from $\mathbb{C}/\Lambda$ to $\tilde{R}$ where $\Phi(0) = (\infty, \infty)$.

Proof.

We have hol. maps.

① $$\mathbb{C}/\Lambda - \{0\} \xrightarrow{\;u \overset{\Phi}{\mapsto} (\wp(u), \wp'(u))\;} \mathbb{C}^2$$

$$u \longmapsto (\wp(u), \wp'(u))$$

② $$\begin{tikzcd} & \tilde{R} - \{0\} \arrow[dr] & \\ \mathbb{C}/\Lambda - \{0\} \arrow[ur] \arrow[rr, "\Phi", "u \mapsto (\wp(u), \wp'(u))"'] & & \mathbb{C}^2 \end{tikzcd}$$

By the cor.

③ The function Φ extends to

$$\mathbb{C}/\Lambda \longrightarrow \mathbb{C}_\infty \times \mathbb{C}_\infty$$

taking 0 to (∞, ∞)

since each coordinate is a meromorphic function and 0 is a pole for $\wp$ and $\wp'$.

④ So we have

$$\begin{tikzcd} & \tilde{R} \arrow[dr] & \\ \mathbb{C}/\Lambda \arrow[ur, "\varphi"] \arrow[rr, "\Phi"] & & \mathbb{C}_\infty \times \mathbb{C}_\infty \end{tikzcd}$$

φ is continuous and holomorphic on $\mathbb{C}/\Lambda - \{0\}$ so φ is actually holomorphic on $\mathbb{C}/\Lambda$

(Problem.)

⑤ Want to show that φ is a hol. isomorphism.

Suffices to show $\deg(\varphi) = 1$.

$$\begin{tikzcd} & \tilde{R} \arrow[dr] & \\ \mathbb{C}/\Lambda \arrow[ur, "\varphi"] \arrow[rr, "\Phi"] \arrow[dr] & & \mathbb{C}_\infty \times \mathbb{C}_\infty \arrow[dl, "\pi_z"] \\ & \mathbb{C}_\infty & \end{tikzcd}$$

or

$$\begin{tikzcd} \mathbb{C}/\Lambda \arrow[r, "\varphi"] \arrow[dr, "P"', "\deg=2"] & \tilde{R} \arrow[d, "\pi_z \circ L \quad \deg=2"] \\ & \mathbb{C}_\infty \end{tikzcd}$$

$\deg \varphi \cdot \deg \pi_z \circ L = \deg P.$ (Problem)

Now $\pi_z|\tilde{R}$ is a covering map of degree 2 (away from zeros of w) so its degree is 2.

We have shown $\deg P = 2$.

$\deg \varphi \cdot 2 = 2.$

So $\deg \varphi = 1.$

$$\begin{array}{c} R \\ \downarrow \pi_z \circ L \leftarrow \deg = 2 \\ \mathbb{C} \end{array}$$

For a generic $z \in \mathbb{C}$ there are 2 values of w so that $w^2 = P(z)$

$(w, -w)$.

We can think of $\wp(z)$ and $\wp'(z)$ as being the analogues of $\sin(z)$ and $\cos(z)$ for R instead of $z^2+w^2=1$ in that

$\cos = \sin'$ and $\sin^2+\cos^2=1$

$$\wp(z) \sim \sin(z) \quad ?$$
$$\wp'(z) \sim \cos(z) \quad ?$$

Analogue of $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \arcsin$

Recall that the elliptic integral (after which the elliptic curve is named) corresponds to

$$\int \frac{dz}{\sqrt{P(z)}} = \int \frac{dz}{\sqrt{z^3+g_1 z+g_2}}$$

On R this became $\displaystyle\int \frac{dz}{w}$.

Prop. $\phi^*\left(\frac{dz}{w}\right) = du$

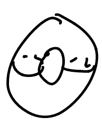
Proof. $\phi^*\left(\frac{dz}{w}\right) = \phi^*(dz) \cdot \frac{1}{\phi^*(w)}$

$$= d\,\phi^*(z) \cdot \frac{1}{\phi^*(w)}$$

$$= \frac{d\,P(u)}{P'(u)} = \frac{\frac{d}{du}P(u)\,du}{P'(u)}$$

$$= \frac{P'(u)\,du}{P'(u)}$$

$$= du.$$

 $\xrightarrow{\Phi} R \subset \mathbb{C}^2$

$\Phi^*\left(\frac{dz}{w}\right) = du$ $\qquad \frac{dz}{w}$

How do we evaluate path integrals for du on $\mathbb{C}/\Lambda$?

(This discussion now appears at the end of the next lecture.)