Riemann Surfaces Lecture 1.

Let $U \subset \mathbb{C}$ be an open set

$$f : U^{\subset \mathbb{C}} \longrightarrow \mathbb{C} \qquad f = u + iv$$

is holomorphic if it satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

We define $f' = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$

$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Our goal in this course is to define Riemann surfaces and do some analysis on them.

following Beardon, A Primer on Riemann Surfaces 14 outline.

There is a second related course taught by Miles Reid which will focus more on an algebraic approach.

Analysis in $\mathbb{R}^3$ can be described in terms of div, curl, grad but it will be more useful for us (the most natural language) to use the language of differential forms. I am not assuming that you know this language so I will start by giving a relatively brief summary and referring you to other sources for a more leisurely introduction.

Forms are useful theoretically and they are useful for specific calculations.

For those of you who haven't seen this before it will be necessary to do some calculations to get a feel for how this works. Suggested references available on YouTube.

Multivariable Mathematics: Linear Algebra, Multivariable Calculus and Manifolds.

Ted Shifrin Math 3510 Day X YouTube.

Days 24 - 29

A k-form on $\mathbb{R}^n$ is an object which can be integrated over a k-dimensional (oriented) submanifold of $\mathbb{R}^n$. It is an object on which integration is natural and well defined.

In particular a 1-form is related to line integrals.

If θ is a k-form then at each point p of $\mathbb{R}^n$ θ_p is a function from k-tuples of vectors to $\mathbb{R}$.

$\theta_p[v_1 \ldots v_k]$ is a number.

"a k-form eats k-tuples of vectors and spits out numbers."

Examples of k-forms.

a multi-index

Let $I : \{1 \dots k\} \to \{1 \dots n\}$ be a function write $I = (i_1 \dots i_k)$.

Define a k-form which we call dx_I as follows:

$dx_{i_1 \dots i_k}$ $\boxed{\text{see}}$ $\boxed{\text{day 24}}$

$dx_I(v_1 \dots v_k)$

$$\begin{vmatrix} v_{1 i_1} & & v_{k i_1} \\ v_{1 i_2} & & \vdots \\ \vdots & & \vdots \\ v_{1 i_k} & & v_{k i_k} \end{vmatrix}$$

$$\begin{array}{c} \text{row } 1 \\ \vdots \\ \text{row } n \end{array} \begin{bmatrix} v_{1,1} & \dots & v_{k,1} \\ \vdots & & \vdots \\ v_{1,n} & & v_{k,n} \end{bmatrix} \begin{array}{l} \leftarrow \text{row } I_1 \\ \\ \leftarrow \text{row } I_2 \end{array}$$

Write $I = (i_1 \dots i_k)$

multiindex

$$dx_I(v^1 \dots v^k) = \det \begin{bmatrix} \text{row } I_1 \\ \vdots \\ \text{row } I_k \end{bmatrix} = \begin{vmatrix} \text{row } I_1 \\ \vdots \\ \text{row } I_k \end{vmatrix}$$

$$= \det \begin{bmatrix} v_{1,i_1} & \cdots & v_{k,i_1} \\ v_{1,i_2} & \cdots & v_{k,i_2} \\ & \vdots & \\ v_{1,i_k} & \cdots & v_{k,i_k} \end{bmatrix}$$

$I : \{1,2\} \to \{1,2,3\}$

$I = (2, 1)$

Example: 2-form in $\mathbb{R}^3$.

$$(v^1, v^2) = \begin{bmatrix} v_1^1 & v_1^2 \\ v_2^1 & v_2^2 \\ v_3^1 & v_3^2 \end{bmatrix}$$

$I(1) = 2$

$I(2) = 1$

$$dx_I(v^1 v^2) = \det \begin{bmatrix} v_2^1 & v_2^2 \\ v_1^1 & v_1^2 \end{bmatrix} \begin{array}{l} \leftarrow \text{row } 2 \\ \leftarrow \text{row } 1 \end{array}$$

I gives a recipe for forming a $k \times k$ minor of the matrix of coordinates of $v_1 \ldots v_k$.

$I'(1) = 1$ $\quad I' = (1,2)$, $\quad \begin{matrix} v_{1,1} & \cdots & v_{k,1} \\ \vdots & & \vdots \\ v_{1,2} & \cdots & v_{k,2} \end{matrix}$

$I'(2) = 2$

$$\begin{array}{ll} dx_{I'}(v^1\, v^2) = \det \begin{bmatrix} v^1_1 & v^2_1 \\ v^1_2 & v^2_2 \end{bmatrix} & \begin{matrix} \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \end{matrix} \end{array}$$

note $dx_{I'} = -dx_I$.

1-form in $\mathbb{R}^2$

$I(1) = 1 \quad \rightsquigarrow (i_1) = (1)$.

$dx_I(v) = v_1$.

Convention: We call this 1-form dx_1.

Note that the role of the indices $1 \ldots n$ is to refer to coordinate functions on $\mathbb{R}^n$. If we have other names for the coordinate functions we can use these, writing dx and dy on $\mathbb{R}^2$ instead of dx_1 and dx_2.

Def. A k-form in $\mathbb{R}^n$ is an expression

$$\theta_p = \sum_{\substack{I:\{1\dots k\}\\ \to\{1\dots n\}}} f_I(p)\,dx_I$$

(we usually assume that f is smooth since we will want to discuss the derivatives of f.)

Note that

$$\overset{i_j = i_k}{\text{If } I(j) = I(k)} \text{ then } dx_I = 0.$$

(If 2 <u>rows</u> are equal then the determinant is zero.)

$\theta_p(v^1 \dots v^k)$ is linear in each variable.

If we switch two vectors then the sign of θ_p changes.

(Behavior of the determinant on columns.)

We make the convention that a 0-form is just a function.

$\theta = f$ (since it depends on 0 vectors)

Prop. θ can be written as $\sum_{I} f_I \, dx_I$, $I : \{1 \dots k\} \to \{1 \dots n\}$

where I is injective and increasing.

Cor. An n-form in $\mathbb{R}^n$ can be written as

$$\theta = f_I \, dx_I$$ where I is the identity map.

We call such a form a volume form.

Cor. If $k > n$ then a k-form in

$\mathbb{R}^n$ is zero.

There is a product on the space of forms which makes it an <u>algebra</u> (over the ring of smooth functions, in fact a graded algebra)

We define the wedge product of a ℓ-form dx_I and a m-form dx_J to be the $\ell+m$-form $dx_{I,J}$ where the multi-index I,J is $(i_1 \dots i_\ell\, j_1 \dots j_m)$.

(In a more thorough treatment of forms we would define the wedge product in a coordinate free way and we would deduce this from the coordinate free definition.)

Prop. $dx_I \wedge dx_J = (-1)^{\ell m} dx_J \wedge dx_I$.

We define the wedge product in general so that it is distributive with respect to multiplication by functions.

Remark: We can also write dx_I as $dx_{I(1)} \wedge dx_{I(2)} \cdots dx_{I(k)}$.

Convention: If we write the coordinate functions in $\mathbb{R}^2$ as x and y then we write dx and dy for the corresponding 1 forms.

Example: $\alpha, \beta, \gamma, \mu$ are functions of x, y

$$(\alpha\, dx + \beta\, dy) \wedge (\gamma\, dx + \mu\, dy)$$

$$= \alpha\, dx \wedge \gamma\, dx + \alpha\, dx \wedge \mu\, dy + \beta\, dy \wedge \gamma\, dx + \beta\, dy \wedge \mu\, dy$$

$$= \alpha\mu\, dx \wedge dy + \beta\gamma\, dy \wedge dx$$

$$= (\alpha\mu - \beta\gamma)\, dx \wedge dy.$$

Reference from Shifrin:

time: 24:00 Wedge Day 25

Exterior derivative. If f is a 0-form:

$$d(f) = \sum \frac{\partial f}{\partial x_j} \cdot dx_j \quad \text{is a 1-form.}$$

Note $d(f)[v] = Df_v$ so the exterior derivative for functions can be identified with the derivative of a function.

Note that both of these things are naturally identified with the <u>row</u> vector of length n:

$$\left[\frac{\partial f}{\partial x_1} \quad \cdots \quad \frac{\partial f}{\partial x_n} \right].$$

since a row vector is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^1$.