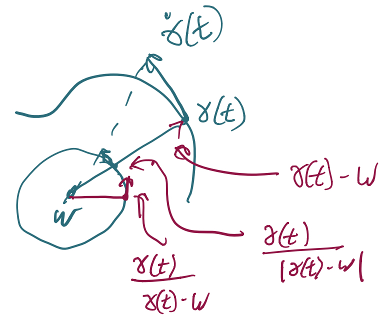
Local versus global questions about path integrals. These raise

There may be global obstructions.

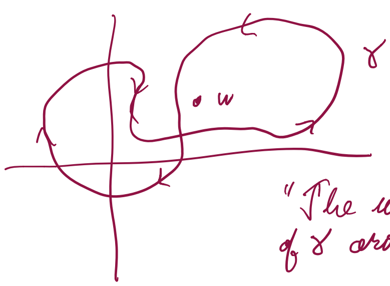


We see one of these in the definition of the winding number: Let γ be a parametrized path in $\mathbb{C} - \{w\}$.

Define:

$$n(\gamma, w) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-w}$$

$\operatorname{Im}\left(\frac{\dot\gamma(t)}{w-\gamma(t)}\right)$ is the speed at which $\gamma(t)$ is rotating around w measured in radians



"The winding number of γ around w."

Re

"Radial component of the velocity."

Lemma

Connection between path integration and path lifting.

$$
\begin{tikzcd}
 & \mathbb{C} \arrow[d, "\exp"] \\
{[a,b]} \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & \mathbb{C}-\{0\} \arrow[u, dashed, bend right=60, "\text{branch of ``log''}"']
\end{tikzcd}
$$

Say $\gamma(a) = z_0$. Let w_0 be a point in $\mathbb{C}$ with $\exp(w_0) = z_0$ (w_0 is a lift of z_0).

For $t \in [a,b]$ define

$$\tilde{\gamma}(t) = w_0 + \underbrace{\int_a^t \frac{\gamma'(s)}{\gamma(s)}\,ds}_{\text{path integral along along } [a,t].}$$

Claim that

$$\exp(\tilde{\gamma}(t)) = \gamma(t).$$

Differentiate the ratio:

$$\frac{d}{dt}\, \gamma(t)/\exp(\tilde{\gamma}(t)) = \frac{d}{dt}\, \gamma(t) \exp\left(-w_0 - \int_a^t \frac{\gamma'(s)}{\gamma(s)}\,ds\right)$$

$$= \gamma'(t)\exp\left(-w_0 - \int_a^t \frac{\gamma'(s)}{\gamma(s)}\,ds\right) + \gamma(t)\exp\left(-w_0 - \int_a^t \frac{\gamma'(s)}{\gamma(s)}\,ds\right)\left(-\frac{\gamma'(t)}{\gamma(t)}\right)$$

$= 0$. So the ratio is constant.

$\exp(\overset{w_0}{\tilde{\gamma}(a)}) = \overset{z_0}{\gamma(a)}$ by assumption so

$\gamma(t)/\exp(\tilde{\gamma}(t)) = 1$.

Corollary. $n(\gamma, w) \in \mathbb{Z}$.

Since γ is a loop $\exp(\tilde{\gamma}(a)) = \exp(\tilde{\gamma}(b)) \Rightarrow \dfrac{\exp(\tilde{\gamma}(a))}{\exp(\tilde{\gamma}(b))} = 1$

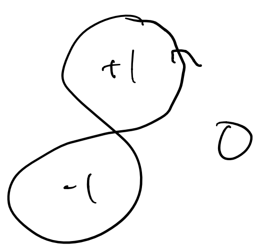
$\Rightarrow \exp(\tilde{\gamma}(a) - \tilde{\gamma}(b)) = 1 \Rightarrow \tilde{\gamma}(a) - \tilde{\gamma}(b) \in 2\pi i \mathbb{Z} \quad \Rightarrow \int_\gamma \frac{dz}{z} \in 2\pi i \mathbb{Z}$

$\Rightarrow n(\gamma, w) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z} \in \mathbb{Z}$.

Corollary. (why?) $n(\gamma, w)$ is constant in each component of the complement of γ.

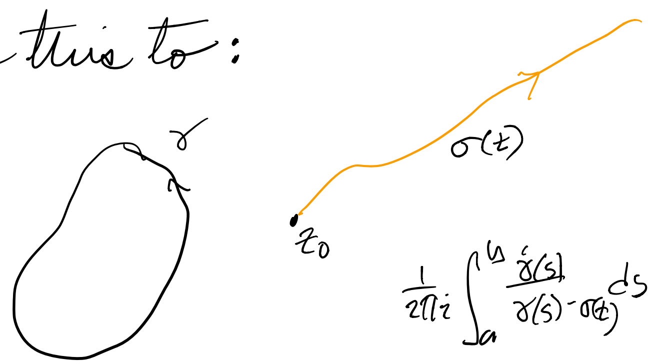
Proof. $\frac{1}{2\pi i}\int_\gamma \frac{dz}{(z-w)} = \frac{1}{2\pi i}\int_a^b \frac{\dot\gamma(s)}{\gamma(s)-w}\,ds$ is a continuous function of w since the integrand varies uniformly continuously. A continuous integral valued function is constant.

Example:



Lemma. If w is in an unbounded path component of γ then $n(\gamma, w) = 0$.

Compare this to:

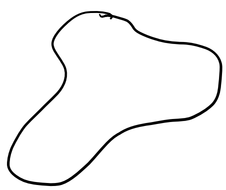


$\lim_{t\to\infty} |\sigma(t)| = \infty.$

$\frac{1}{2\pi i}\int_a^b \frac{\dot\gamma(s)}{\gamma(s)-\sigma(t)}\,ds \to 0$ as $t \to \infty$

How do we define the "inside" of a curve?

Can do it in terms of the winding number.



Theorem. A simple curve cuts the plane into two components. One is bounded and one is unbounded.

(Proof is strictly topological. One way uses Van Kampen's theorem.)

Winding number of γ around a point in the bounded component is ± 1.

By switching the orientation if necessary we may assume that the winding # is $+1$. (Say γ is oriented in an anti-clockwise direction.)

We say that w is <u>inside γ</u> if $n(\gamma, w) = 1$.

If f is holomorphic in a domain D then f has a Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

valid in a disc $\{z : |z-z_0| < r\}$ contained in D.

Lemma.

If f is not constant in D then for every choice of z_0 in D there is some $n \geq 1$ such that $a_n \neq 0$. This is because the set of z_0 for which $a_1 = a_2 = \dots = 0$ is both open and closed in D.

It is closed since each a_n is a continuous function. It is open since if all $a_1 = a_2 = \dots = 0$ then f is constant in a neighborhood hence all higher coef. vanish in that nbd.

Either f is locally constant at <u>every point</u> or at <u>no point.</u> If it is locally constant at every point it is constant by connectivity.

If f is not constant
We can now write

$$f(z) = f(z_0) + (z-z_0)^N \left(a_N + a_{N+1}(z-z_0) + \dots \right)$$

with $a_N \neq 0$. By continuity there is some positive r_1 such that $f(z) \neq f(z_0)$ when $0 < |z-z_0| < r_1$.

Let D be a domain in $\mathbb{C}$.

Prop. If f is holomorphic and not constant in D then the solutions of $f(z) = w$ are isolated points of D.

Cor. The ring of holomorphic functions on a domain D is an integral domain.

Injective map into the field of fractions.

Follows that we can create a field by inverting non-zero elements. These inverted non-zero elements will correspond to meromorphic functions.

A point w is an isolated singularity of f if f is holomorphic in some domain $\Delta = \{z : 0 < |z-w| < r\}$.

An isolated singularity is either

1) removable if $\lim_{z \to w} f(z)$ exists.

In this case
f extends to a holomorphic function in Δ.

2) a pole if $\lim_{z \to w} |f(z)| = \infty$

in this case $f(z) = \frac{g(z)}{(z-w)^N}$ with g hol. and $g(w) \neq 0$.

3) an essential singularity.

If all singular pts are poles then f is meromorphic.

Jordan Curve Theorem. The bounded component is a disk.

(Can be proved by modifying the Riemann mapping theorem.)

Obstruction to having an anti-derivative is a non zero homomorphism from $\pi_1(U) \to \mathbb{Z}$

$\mathbb{Z}$ or $\mathbb{C}$?

Thm. If f is a meromorphic function in D and if γ is a Jordan curve in D such that

(1) no poles of f lie on γ and

(2) the points inside γ also lie in D

then
$$\int_\gamma f = 2\pi i \sum \mathrm{Res}_f(p_j)$$
where p_j are the poles of f inside γ and $\mathrm{Res}_f(w)$ is the residue of f at w.

Here $\mathrm{Res}_f(p_j)$ is the coef. of z^{-1} in the Laurent expansion of f or alternatively $\int_\gamma f\,dz$ for γ a small anti-clockwise oriented loop around p_j.