Recall that if f is not locally constant then $f(z) = a_0 + \dots a_k z^k + a_{k+1} z^{k+1} \dots$ $a_k \neq 0$, $k = \nu_f(0)$.

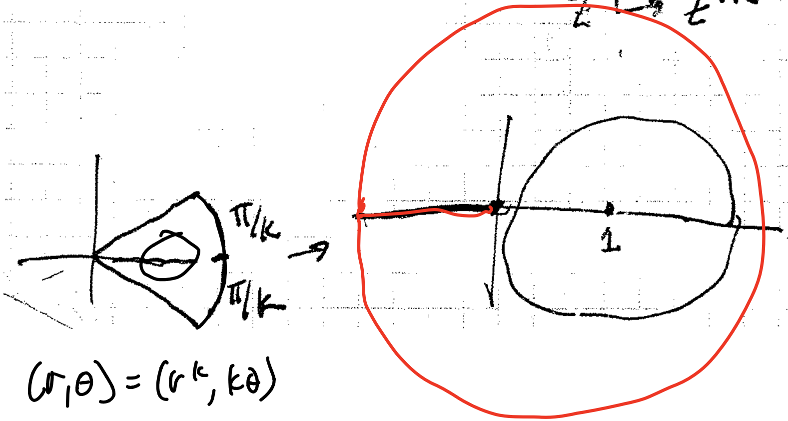
Lemma. Let f be a holomorphic function on an open nbd. U of 0 in $\mathbb{C}$ with $f(0) = 0$ but f not identically 0. ($\nu_f(0)$ defined and finite) Then there is a disk $D \subset U$ centered at 0 and

a holomorphic function g with $g'(0) \neq 0$ ($\nu_g(0) = 1$) and $f(z) = h^k(z)$ on D where $k = \nu_f(0)$.

Proof. $f(z) = a_k z^k + a_{k+1} z^{k+1} + \dots$ $a_k \neq 0$
$a_0 = f(0) = 0$, $a_1 \dots a_{k-1} = 0$.

$$f(z) = a_k z^k \underbrace{(1 + b_1 z + b_2 z^2 + \dots)}_{g(z)} \quad \text{where } b_j = \frac{a_{k+j}}{a_k}.$$

Assume $D \subset U$ is small enough that $|\sum b_j z^j| < 1$ then the image of $z \mapsto (1 + b_1 z + b_2 z^2 + \dots)$ is contained in a disk in $\mathbb{C} - \{0\}$. In particular we can choose a branch of $z \mapsto z^{1/k}$ in this disk and define



$(r, \theta) = (r^k, k\theta)$

$$h(z) = a_k^{1/k} z \, g^{1/k}(z)$$

$(r^{1/k}, \theta/k) \mapsto (r, \theta)$

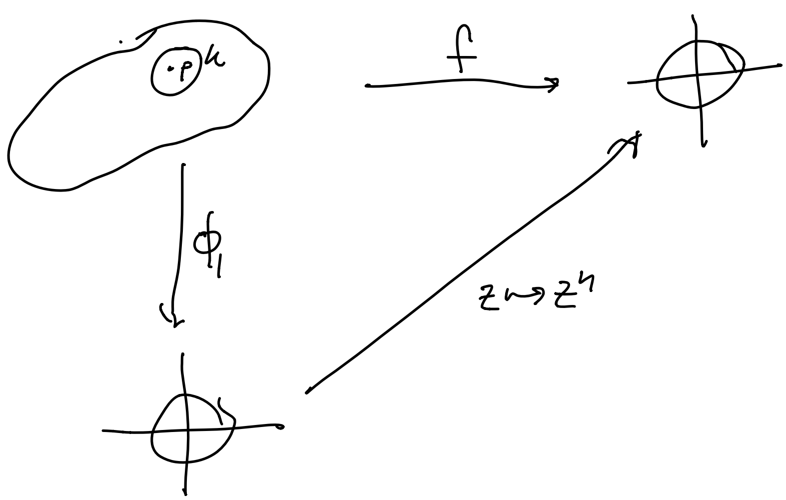
Let $h(z) = a_k^{1/k}\, z\, g(z)$ for some choice of k-th root

then $h^k(z) = a_k z^k (1 + b_1 z + b_2 z + \dots) = f(z)$.

Furthermore since $h'(0) = a_k^{1/k} \neq 0$.

Theorem. Let f be a hol. $\mathbb{C}$ valued function in a Riemann surface R. Let $p \in R$ then there is a locally invertible function ϕ_1: nbd of p to $\mathbb{C}$ so that $f = \phi_1^k$ ($f(w) = \phi_1^k(w)$.)

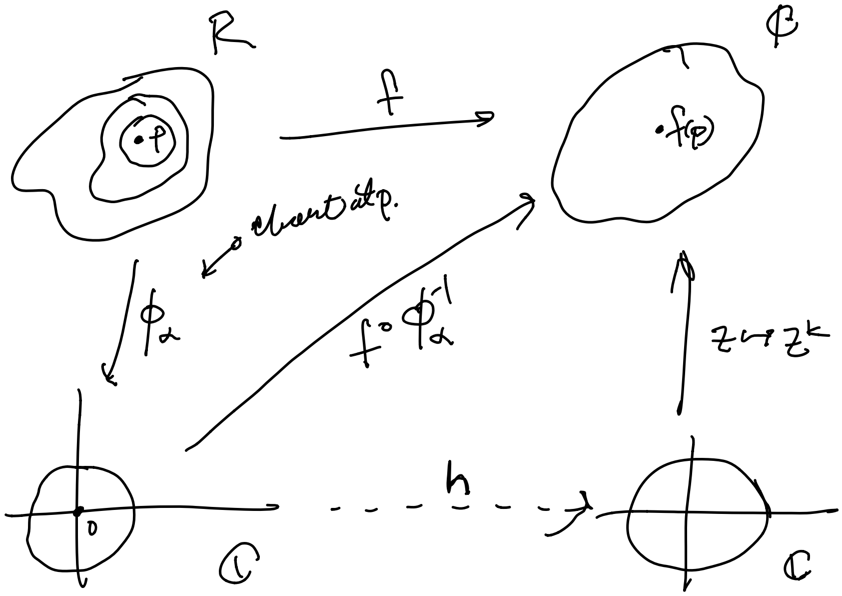
Theorem. Let f be a non-constant complex valued holomorphic function on a connected Riemann surface R. Let $p \in R$ then there is an invertible holomorphic function $\phi_1 : \underset{\ni p}{U} \to \underset{\ni 0}{V}$ so that $f(z) = \phi_1^n(z)$ where $n = v_f(p)$.



(We can think of ϕ_1 as a chart.)

Proof.

Adjust ϕ_α by adding a const so $\phi_\alpha(p)=0$.
In the atlas? Could be.

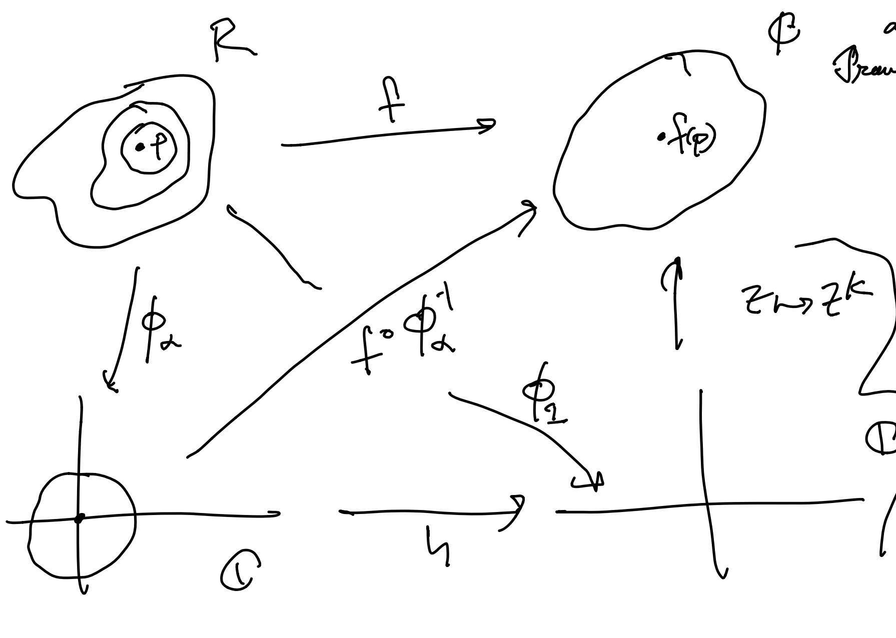


Apply Lemma to get h.

Recall h has valence 1 at 0. Thus h is locally invertible.

Let $\phi_1 = h\circ\phi_\alpha$.

ϕ_1 is holomorphic and bijective.
ϕ_1 __could__ be in the atlas $\mathcal{A}$.
Throw it in create atlas $\mathcal{A}'$.
Transition fcn. with ϕ_α is hol.



and is locally invertible by the inverse fcn. thm.

$f=\phi_1^k$ near p.

Theorem. Geometric formulation.

If $f: R \to S$ is a holomorphic map between Riemann surfaces and $p \in R$ then there are charts ϕ_α with $\phi_\alpha(p) = 0$ and ψ_β with $\psi_\beta(f(p)) = 0$ where

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "\phi_\alpha"] \arrow[dr, "\psi_\beta \circ f"] & S \arrow[d, "\psi_\beta"] \\ \mathbb{C} \arrow[r, "z \mapsto z^k"] & \mathbb{C} \end{tikzcd}$$

with $k = v(f, p)$.

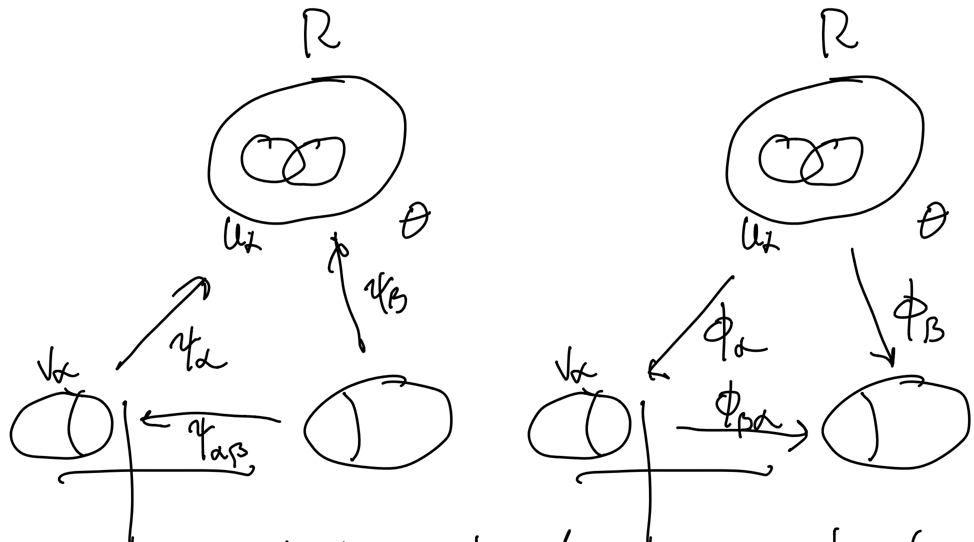
Proof. There is some chart ψ_1 defined in V_1 with $f(p) \in V_1$. Let $\psi_\beta(q) = \psi_1(q) - \psi_1(f(p))$.

Now consider $\psi_\beta \circ f$ and find a chart ϕ_α in which this function can be written as $z \mapsto z^k$.

For the next discussion it is useful to talk about an atlas of inverse charts.

Say $\phi_\alpha : U_\alpha \subset R \longrightarrow V_\alpha \subset R$ is a chart in $\mathcal{A}$ where $\alpha \in A$.

Let $\mathcal{A}^{-1}$ consist of   Let $\psi_\alpha : V_\alpha \longrightarrow U_\alpha$ be inverse charts $\psi_\alpha = \phi_\alpha^{-1}$



$\psi_\alpha \circ \psi_{\alpha\beta} = \psi_\beta$ where defined. $\phi_{\beta\alpha} \circ \phi_\alpha = \phi_\beta$

Set $\psi_{\alpha\beta} = \phi_{\beta\alpha}^{-1}$.