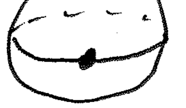
Topological invariants of closed surfaces.

Every compact orientable surface is homeomorphic to a sphere, torus or a connected sum of g tori. We define the connected sum by removing disks from both surfaces and gluing along the resulting boundaries.

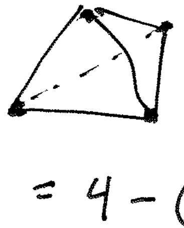
$S^2, T^2, T^2 \# T^2, T^2 \# T^2 \# T^2, \ldots$

We say that the g is the genus. ~~thus $g(S^2)=0$, $g(T^2)=1$~~
The genus is a complete invariant in this context.
Also useful to consider the Euler characteristic $\chi(M)$.
Compute this by taking a finite triangulation of M.
$\chi(M) = \#\text{vertices} - \#\text{edges} + \#\text{faces}$.
This works even if the "faces" are simply "cells" and the edges do not have distinct endpoints.



$\chi(S^2) = 1 - 1 + 2 = 2$

Also $\chi(M) = \operatorname{rank} H^0 - \operatorname{rank} H^1 + \operatorname{rank} H^2$
so $\chi(M)$ is a topological invariant.



$= 4 - 6 + 4 = 2.$

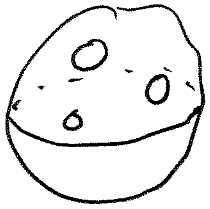
Topological invariants of closed surfaces with boundaries.

Def. If M and N are closed surfaces with boundaries then the connected sum $M \# N$ is the result of removing disks from M and N and gluing them together along the boundaries of M, N.

M $\qquad$ N $\qquad$ $M \# N$

Thm. Every connected ~~closed~~ compact surface without boundary is the S^2, T^2, ~~or~~ the connected sum of g copies of T^2 or one of these with ~~Thm. Every~~ n open disks removed.
g is the genus, n is the # of boundary components.
The proof can be approached by induction. Using Morse theory we can see that every ~~surface in~~ ~~co~~ such surface can be built by attaching "handles" to a disk and possibly

Genus according to Riemann: genus is the maximal number of ~~simples~~ disjoint simple closed curves we can remove without disconnecting the surface.



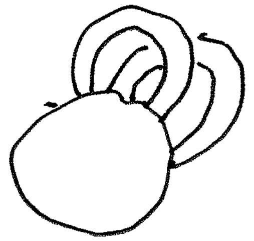
Genus can also be understood in terms of intersection form on $H_1^*(M;\mathbb{Z})$.

~~Intersection form on $H_1(M;\mathbb{Z})$~~

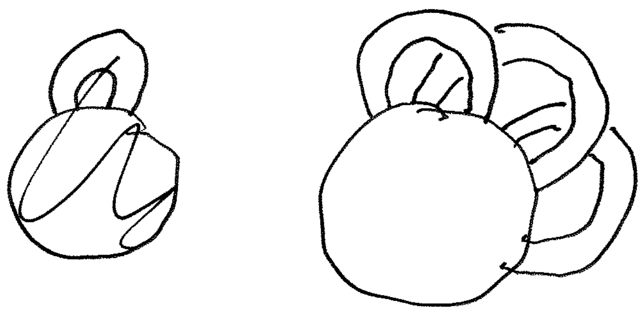
filling in some boundary components.

Example:

S^2 - 2 disks $\quad g=0 \quad n=1$



T^2 - 1 disk $\quad g=1 \quad n=1$



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A convenient invariant is the Euler characteristic. Defined in terms of triangulations or as an alternating sum of ranks of homology groups

$$\chi(M) = \sum (-1)^i \dim H^i(M;\mathbb{Q}).$$

Second definition shows that it is a topological invariant, in fact an invariant of the homotopy type of M. Defined in all dimensions.

Example: $\chi(D^n) = \chi(\text{pt.}) = 1$

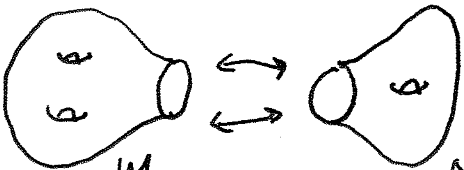
$\chi(S^1) = 0$

$\chi(S^2) = 2.$

~~Prop. $\chi(M \# N) = \chi(M) + \chi(N) - 2.$~~

~~Proof. If we join 2~~

Proof. If $R = M \cup_{S^1} N$ where we join M and N along a circle then $\chi(R) = \chi(M) + \chi(N)$.



M N

Proof. Triangulate the boundaries with n vertices and n edges. ~~The~~ Extend the triangulations to M and N. Counting cells we have $\chi(M \cup N) = \chi(M) + \chi(N) - \chi(S^1)$

$= \chi(M) + \chi(N).$

Prop. $\chi(M \# N) = \chi(M) + \chi(N) - 2$.

Proof. $\chi(M - D) + \chi(D) = \chi(M)$
$\chi(N - D) + \chi(D) = \chi(N)$.

$$\chi(M \# N) = \chi(M - D) + \chi(N - D) = \chi(M) + \chi(N) - 2.$$

~~If M is~~

Prop. If M is a ~~surfa~~ compact surface with genus g and n boundary components then $\chi(M) = 2 - 2g - n$.

Proof. If M has no ∂ components then $M = T^2 \# T^2 \# \dots T^2$. Taking connected sum $g-1$ times yields $\chi(M) = g \cdot \chi(T^2) - \cancel{g}\, 2(g-1) = 2 - 2g$.

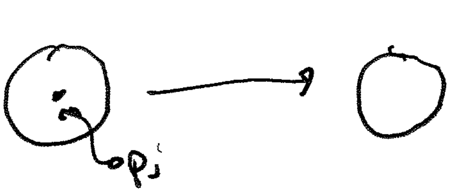
As above removing a disk reduces $\chi(M)$ by 1 and adds 1 boundary component.

Prop. Let $f: M - \{\bar{p}_1 \dots \bar{p}_m\} \to N - \{\bar{q}_1 \dots \bar{q}_m\}$ be a proper, non-constant holomorphic map between connected Riemann surfaces with M, N compact. Then f extends to $F: M \to N$. Furthermore

$F^{-1}(\{\bar{q}_1 \dots \bar{q}_m\}) = \{\bar{p}_1 \dots \bar{p}_m\}$.

Proof. Apply previous proof. Choose nbds V_r of q_r and U_s of p_s. $N - \cup V_r$ is compact. Use properness to ~~see~~ that $f^{-1}(N - \cup V_r)$ is compact. After cutting down the nbds. U_s to U_s' we see that each maps into a nbd. V_r. f is ~~and~~ holomorphic in $U_s' - p_s$ and bounded so it has a holomorphic extension. Call the extended function F. Since F comes from f it is immediate that $F(\{\bar{p}_1 \dots \bar{p}_m\})$ ~~$\subset \{\bar{q}_1 \dots \bar{q}_m\}$~~ $F^{-1}(\{q_1 \dots q_m\}) \subset \{p_1 \dots p_m\}$

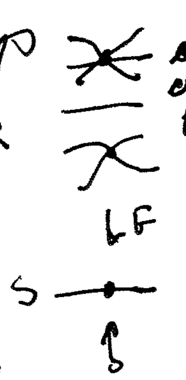
~~(In order for f)~~ Say there is a p_j not in $F^{-1}(\{q_1 \dots q_m\})$ then we violate properness.



① ② ③ + ②

Definitions. $C(F) =$

Any ~~Even~~ $f: R \to S$ is a non-constant proper map between connected Riemann surfaces.



Set of $p \in R$ where $V_f(p) > 1$ we call the critical points of f and denote by $C(f)$.

We call $f(C(f))$ the branch points of f and ~~and~~ denote it by $B(f)$.

Proper map between surfaces of finite type gives

Terminology: Branched cover.

$V_f(p)$ is the multiplicity of the branching at p. degree d is the number of sheets.

We can also define a multiplicity at the punctures.

A proper map $f: R \to S$ between surfaces of finite type has a unique extension to a holomorphic between compact surfaces,

$F: \bar{R} \to \bar{S}$

"punctures of R".

$\bar{R} - R = \{\bar{p}_1 \dots \bar{p}_n\}$

Define $V_f(\bar{p}_j) = V_F(\bar{p}_j)$.

Riemann-Hurwitz Thm. Let $f: R \to S$ be a proper, non-constant holomorphic map of degree d between Riemann surfaces of finite type ~~degree d~~. Then

$$(1) \qquad \chi(R) = d \cdot \chi(S) - \sum_{\substack{p:\, V_f(p) > 1 \\ p \in R}} (V_f(p) - 1).$$

$$(2) \qquad \#\text{punctures of } R = d \cdot \#\text{punctures of } S - \sum_{p \in \bar{R} - R} (V_f(\bar{p}) - 1)$$

$\bar{p}$ a puncture of R with $V_f(\bar{p}) > 1$.

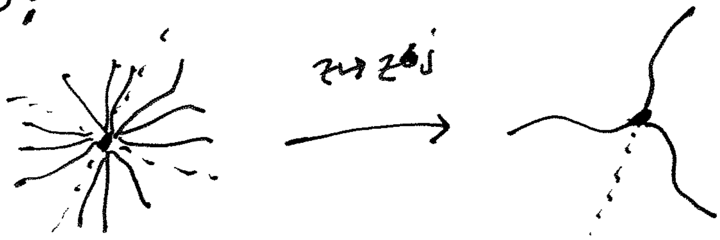
Proof. ~~Note~~ Given $f: R \to S$ we can apply (1) to $f: R \to S$ or $F: \bar{R} \to \bar{S}$. (2) makes sense for $f: R \to S$. "genus" discussions do not distinguish between R, S and $\bar{R}, \bar{S}$, so we can use (1) to analyze the genus of R and S.

Choose a triangulation of ~~of~~ $\bar{S}$ for which the points $\{q_1 \ldots q_r\}$ are included as vertices and the critical values are included as vertices.

We can pull this back to $\bar{R}$. We use the fact that away from $F^{-1}(\sigma(F))$ the map is a covering so the inverse image of any 1-cell or 2-cell is a ~~each~~ disjoint union of 1-cells or 2-cells.

~~What happens~~ also true that the inverse image of any point is a finite union of points.

Near a branch point ~~b~~ in S the triangulation looks like:



So the triangulation ~~...~~ τ of $\bar{S}$ gives a triangulation

Inverse image of the closure of a cell is the closure of a cell upstairs.

Now we can pass from $\bar{R}$ and $\bar{S}$ to R and S by throwing away certain vertices.

The alternating sum formula still calculates the Euler characteristic since removing a point reduces χ by 1 and reduces the alternating sum by 1.

$$\chi(R) = \#V(R) - \#e(R) + \#f(R) \qquad \chi(S) = \#V(S) - \#e(S) + \#f(S).$$

$$\chi(R) - d\chi(S) = \#V(R) - \#V(S)$$

$$= \sum_{q \in V(S)} \#F^{-1}(q) - d$$

Recall that $\sum_{F(p)=q} v_F(p) = d$

$$= \sum_{q \in V(S)} \sum_{\substack{\cancel{F(p)=q} \\ p:\, F(p)=q}} \left(1 - v_F(p)\right)$$

$$= \sum_{p \in V(R)} 1 - v_F(p)$$

p only contributes to the sum if $v_F(p) > 1$

$$= \sum_{C(F)} 1 - v_F(p).$$