~~Abr~~ Change of direction for last 2 lectures.

More interesting directions to pursue than time. Topology & hol. diff.

Due to falling attendance & deduce that ~~Given that~~ many people are familiar with hyperbolic geometry.

~~Since~~ For those who are not I added 2 problems to the last example sheet that bring Monday's lecture to a conclusion.

~~For those w~~

Switch to a topic on the Course Outline that I skipped over.

Fill in a gap.

Prove a basic fact about hol. differentials.

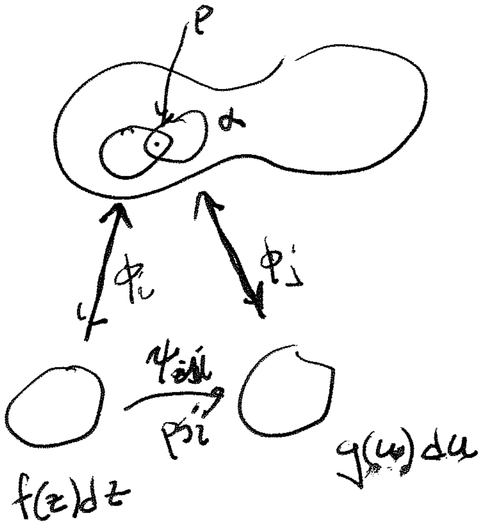
Apply this.

Course Outline 6. Hyperbolic surfaces.

5e. Holomorphic 1-forms and topology.

Degree genus formula.

Recall that a holomorphic 1-form α on R ~~in~~ can be written in terms of local charts as $f(z)dz$. If ~~f~~ the 1-form vanishes at a point in one chart then ~~the~~ the order of vanishing of f is well defined independently of the chart.



$$f(z)dz = \underbrace{\frac{d\psi_{ij}}{dz}} \cdot g(u)du$$

non-zero ~~pt~~ g and f have zeros of the same order.

We have shown how a holomorphic 1-form α on R gives rise to a translation structure on R away from the zeros of α. In particular it determines a "flat" geometry on the surface. What about the zeros?

On the other hand we have seen that we can get ~~translation~~ Riemann surfaces with translation structures by gluing together polygons.

The issue arises at points where the sum of the cone angles is something other than 2π. Call these cone points. (singularity for the translation atlas.)

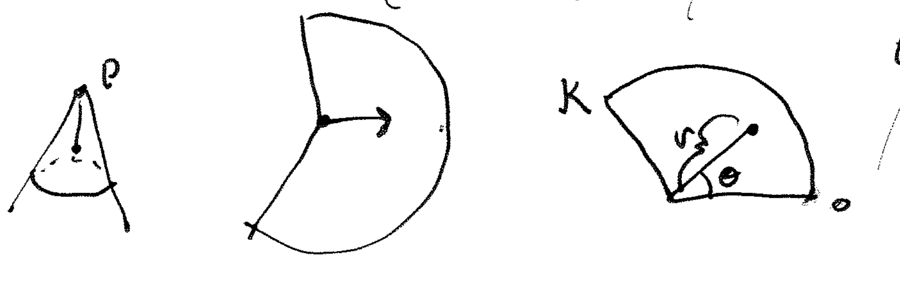
The technique was to introduce "geometric polar coordinates" (r, θ) at the cone point p.

Here r measures distance from p and

θ measures cone angle

$\theta \in [0, K]$ $0 < K$.

can be bigger than 2π.



Now I want to work backwards and show that ~~every~~ a zero of a holomorphic 1-form corresponds to a cone point. In particular ~~we~~ in a nbd. of a zero we can ~~introduce geometric polar coordinates.~~
obtain the geometry by gluing together polygons.

First step.

Prop. If $\alpha = f(z)dz$ is defined in a nbd. of 0 ~~the~~ and ~~or~~ f vanishes to order n at 0 then we can choose a new local coordinate ~~u~~ ~~w~~ u in a nbd. of 0 so that ~~f(z)~~ ~~a~~ ~~then~~ $\alpha = (n+1) u^n du$.
(Pullback of dw under $G(u) = u^{n+1}$)

Proof. Say $f(z) = z^n h(z)$ where $h(0) \neq 0$.

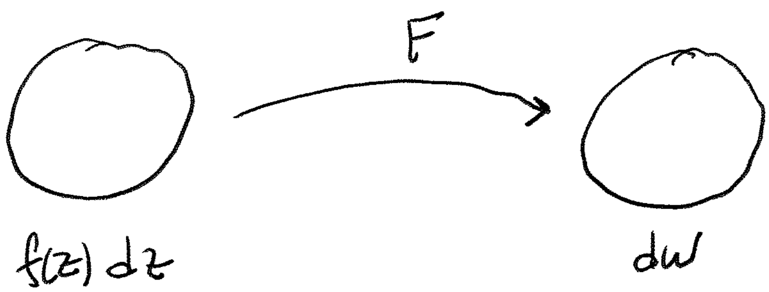
Let $F(z)$ be an anti-derivative near 0.

~~So~~ $F'(z) = f(z)$ and $F(0) = 0$.

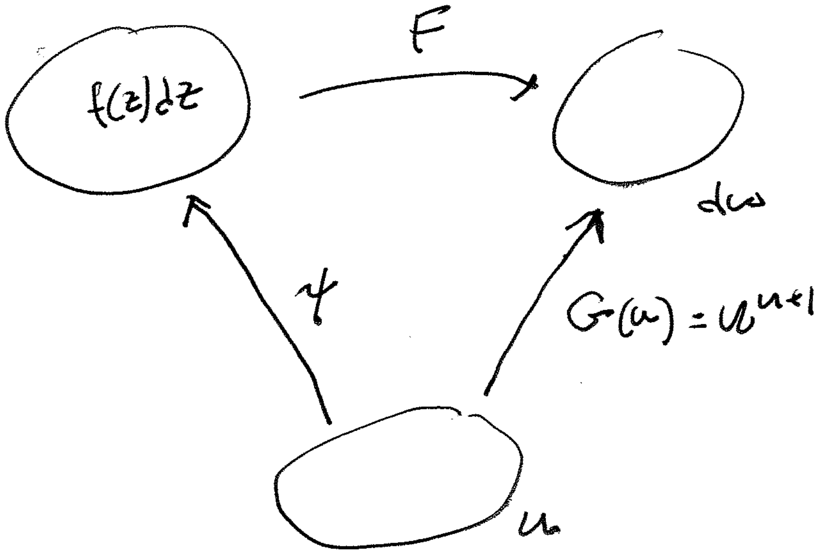
$$f(z) = a_n z^n + a_{n+1} z^n + \dots \qquad F(z) = \frac{a_n z^{n+1}}{n+1} + \frac{a_{n+1} z^{n+2}}{n+2} \dots$$

$$F(z) = z^{n+1} \cdot H(z) \qquad H(0) \neq 0.$$

$w = f(z)$



Now $F^*(dw) = \frac{dF}{dz} \cdot dz = f(z)dz$.



We have already shown that hol. functions ~~are locally~~ can be in standard form. There is ~~a ψ with~~ an invertible ψ so that $F(\psi(u)) = u^{n+1}$. Let $G(u) = u^{n+1}$.

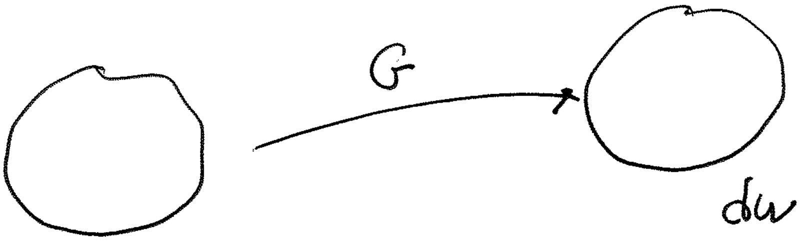
Then $\psi^*(f(z)dz) = G^*(dw) = \frac{dG}{du} u^{n+1} = (n+1)u^n du$.

$$\psi^*(F^*(dw)) = (F \circ \psi)^*(dw)$$

Second step

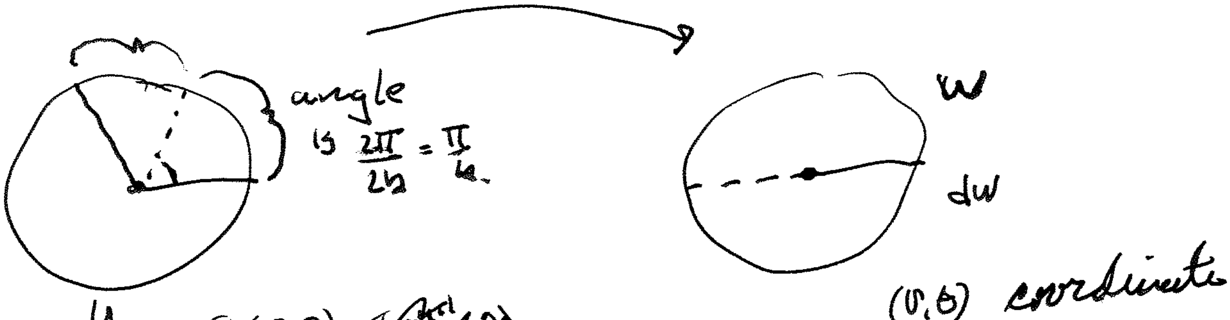
Prop. Let $\alpha = f(z)dz$ be a hol 1-form with a zero of order n then the corresponding translation structure has a cone type singular point with cone angle $2\pi(n+1)$.

Let us now interpret this local form geometrically.



$\alpha = (n+1) u^n du$

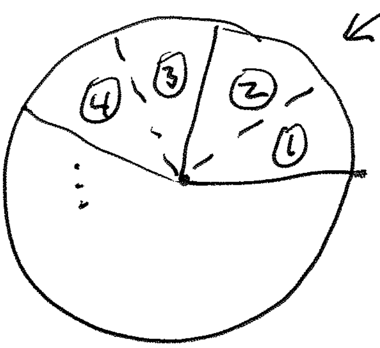
If G where 1-1 we could think of G as giving us a translation structure. We can divide the domain into sectors on which G is 1-1.



$G(r,\theta) = \cancel{(r^{n+1}(n\theta))}$
$= (r^{n+1}, (n+1)\theta)$.

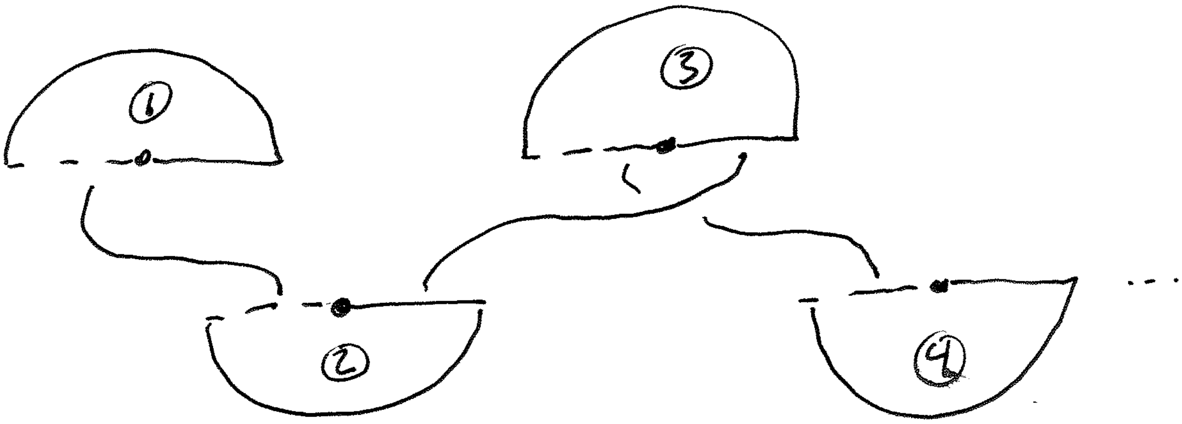
~~Note that the (r,θ) coordinate in the u disk.~~

G is a local isometry from the α geometry on the u disk to the standard geometry on the w disk. Each sector of the α disk gets the geometry of a half-disk. ~~(so that~~



2(n+1) sectors each ~~with~~ geometrically equivalent to a half-~~plane~~ disk.

This picture contains the gluing diagram for these half disks.



This construction builds a cone. ~~The total~~ Each half-disk contributes a cone angle of π. The number of half-disks is ~~4~~ $2(n+1)$ for a total cone angle of $2\pi(n+1)$.

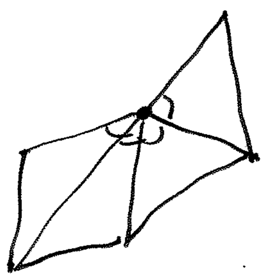
Proposition. Let α be a holomorphic 1-form on a compact surface R, then let p_j be the zeros of α then

$$\chi(R) = -\sum \mathrm{Ord}(p_j).$$

Remove disks of radius r around the zeros.

Boundary contribution is $-$total cone angle.

Proof. Can prove this with Gauss-Bonnet.
Alternate proof. Say that R has a triangulation with vertices at the zeros of α. Let f, e, V be the # of triangles, edges, vertices.
Can add up the angles of the triangles in two ways.

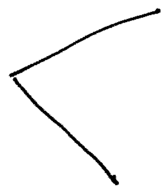


$\sum$ all angles of all triangles $= \sum$ cone angles at vertices

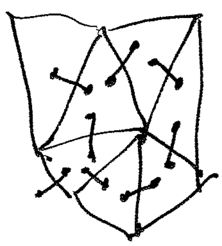
$= \pi \cdot f$

$$\pi f = \sum_{j=1}^{V} (\mathrm{Ord}(p_j) + 1) 2\pi$$

$$f = 2 \sum \mathrm{Ord}(p_j) + 2V$$



Relation between the number of edges and the number of faces: $3f = 2e$.



Now $\chi(R) = f - e + V$

$$2\chi(R) = 2f - 2e + 2V$$
$$= 2f - 3f + 2V$$
$$= 2V - f$$
$$= 2V - \left(2 \cdot \sum \mathrm{Ord}(p_j) + 2V\right)$$
$$2\chi(R) = -2 \sum \mathrm{Ord}(p_j)$$
$$\chi(R) = -\sum \mathrm{Ord}(p_j).$$