$$\phi(z) = \int_{\partial D_2} \frac{w\, f_z'(w)}{f_z(w)}\, dw$$

One variable notation

$$f_z(w) = P(z, w)$$

$$\phi(z) = \int_{\partial D_2} \frac{w}{P(z,w)} \frac{\partial P}{\partial w}\, dw$$

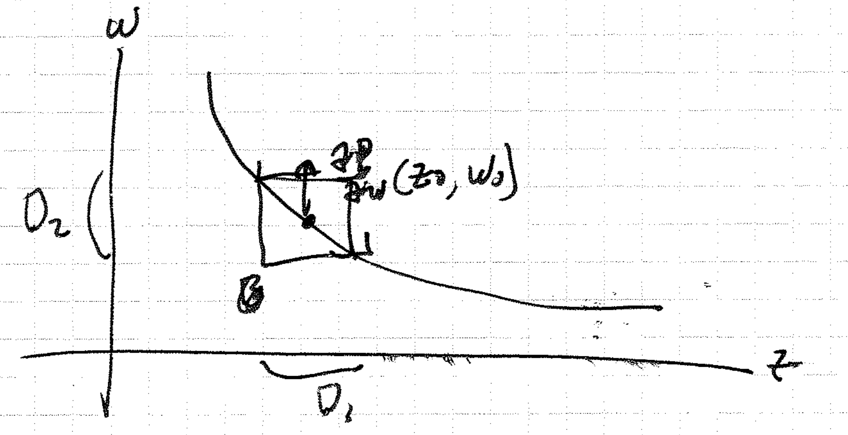
$$= \int_0^1 \frac{w}{P(z,w)} \cdot \frac{\partial P}{\partial w}^{(z,w)} \cdot \gamma'(t)\, dt \qquad \text{where } (z, \varepsilon e^{2\pi i t} + w_0)$$

$$\gamma_z(t) = \varepsilon \cdot e^{2\pi i t} + w_0$$

Want to know that $\phi(z)$ is holomorphic in z. Follows since the integrand is holomorphic in z, and we can differentiate under the integral sign.

$$= \int_0^1 \frac{w(\gamma(t))}{P(\gamma(t))} \frac{\partial P}{\partial w}(\gamma(t)) \cdot \gamma'(t)\, dt$$

It follows from the holomorphic implicit function theorem that we can build charts for $R = \{(z,w) : P(z,w) = 0\}$.



& $R \cap B = \{(z, \phi(z))\}$.
$R \cap (D_1 \times D_2)$

Consider a point $(z_0, w_0) \in R$ where $\frac{\partial P}{\partial w}(z_0, w_0) \neq 0$.

We get a nbd of the pt. by intersecting R with the open set $D_1 \times D_2$. $\quad U = R \cap (D_1 \times D_2)$

We get a chart $\psi = \pi_z|_R : U \to \mathbb{C}_z$ by $\pi_z(z, w) = z$.

We see that π_z is a local homeomorphism since it has a left inverse

$u(z) = (z, \phi(z))$

As $u \circ \pi_z(z, w) = u(z) = (z, \phi(z)) = (z, w)$.

(2 points in $R \cap B$ with same z coord are equal.)

Case of $\frac{\partial P}{\partial z}(z_0, w_0) \neq 0$ we get charts taking values in $\mathbb{C}_w$.

Change of coordinate map has form $\pi_w \circ \phi(z)$
$= \pi_w[(z, \phi(z))] = \phi(z)$.

Historically the study of polynomial equations began with the study of real curves.

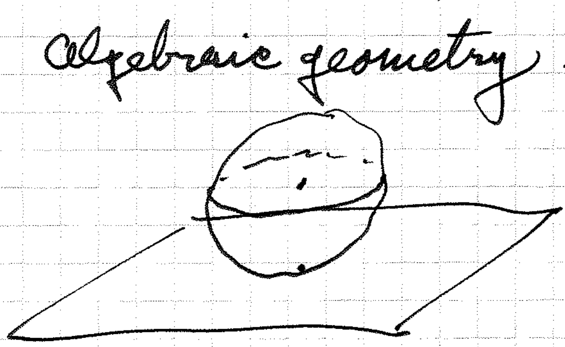
Lines and quadratic equations were studied by the Greeks. Newton studied curves of degree 3. [?81]

Obtained 72 cases. Overlooked 6.
Criticised by Euler and others for not having a guiding principal.

Two simplifying ideas: Consider curves up to projective transformations first and consider complexified curves.
(Newton discusses projective equivalence "method of shadows")
Projective curves.

Let me discuss projective curves in the real setting. Historically first and easier to draw. Construction works over any field.
~~Wonder~~

Algebraic geometry construction. Illustrate with $\mathbb{R}$.



We described the ~~complete~~ 1-pt. completion of $\mathbb{R}^2$ by stereographic projection through the north pole.

We can also form a compaction by ~~show~~

stereographic projection through the center of the earth.

We realize $\mathbb{R}^2$ as the lower hemisphere and we can ~~add~~ complete it by adding the equator.

Note that a point in $\mathbb{R}^2$ corresponds to a line in $\mathbb{R}^3$, ~~a line in $\mathbb{R}^2$ corres~~

If we take a pair of parallel lines in $\mathbb{R}^2$ and follow them in the same direction then the points converge to the same point on the equator.

In this ~~model~~ completion a pair of parallel lines intersect in 2 points on the equator. We want lines to intersect in a single point so we identify antipodal points on the equator.

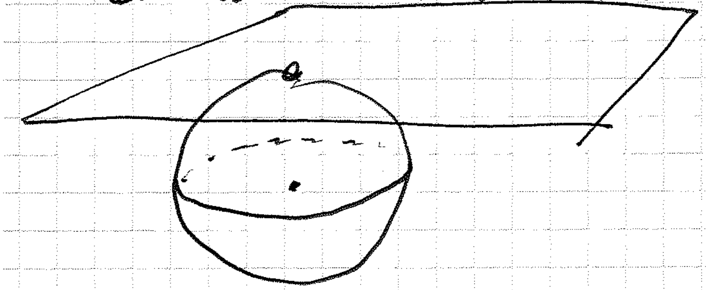
Call the result $\mathbb{RP}^2$, the real projective plane.

In $\mathbb{RP}^2$ lines in $\mathbb{R}^2$ are "completed" to form circles by the addition of a point at ∞.

We would like to similarly complete all algebraic curves.

Points in $\mathbb{RP}^2$ correspond to lines in $\mathbb{R}^3$.

Let me redraw the picture



Plane corresponds to $\{(X, Y, 1)\} \subset \mathbb{R}^3$.

~~Still the case that~~

Let $P(X,Y)$ be a polynomial equation.

We would like to find a polynomial equation $Q(X,Y,Z)$ so that $\{P(X,Y)=0\} = \{(X,Y) : Q(X,Y,1)=0\}$ and the zero set of Q is a union of lines through the origin.

Let $R(X,Y,Z) = P\left(\frac{X}{Z}, \frac{Y}{Z}\right)$

$R(X,Y,Z) = R(\lambda X, \lambda Y, \lambda Z)$ so zero set of R is a union of lines.

If $P = \sum a_{nm} X^n Y^m$ then

$$R(X,Y,Z) = \sum a_{nm} \frac{X^n Y^m}{Z^{n+m}}$$

Note that the largest value of $n+m$ that occurs is $d = \deg(P)$. We can remove denominators by multiplying by Z^d.

$$Q(X,Y,Z) = Z^d R(X,Y,Z) = \sum a_{nm} X^n Y^m Z^{d-(n+m)}.$$

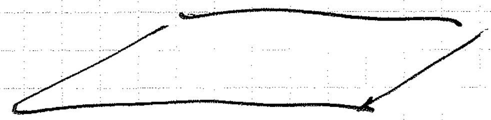
Q is an example of a homogeneous polynomial: all terms have the same degree.

$$Q(\lambda X, \lambda Y, \lambda Z) = \lambda^d Q(X,Y,Z)$$

Q is not constant on lines in general but if Q ~~vanish~~ the set where $Q=0$ is a union of lines.

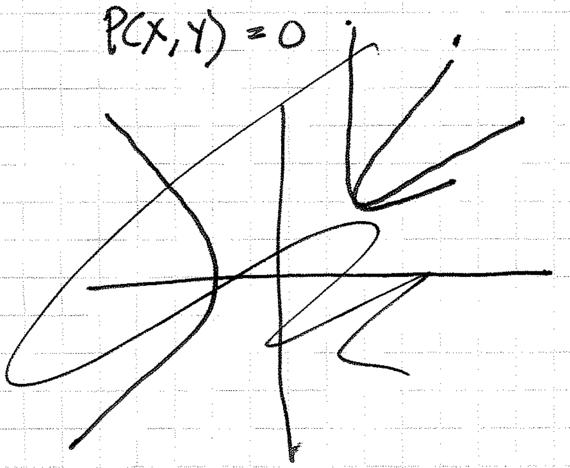
~~Au~~ The set of zeros of a homogeneous polynomial in $\mathbb{RP}^2$ is called a projective curve, An affine curve gives rise to a projective curve. ~~Claim that~~ Claim that

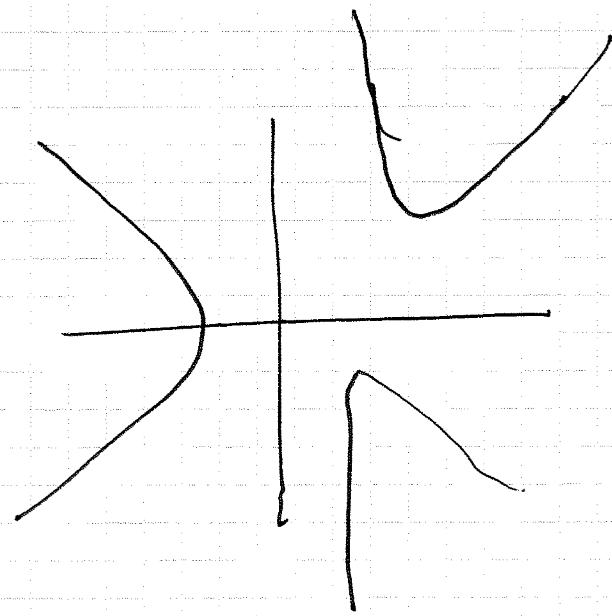
~~Example: R(~~



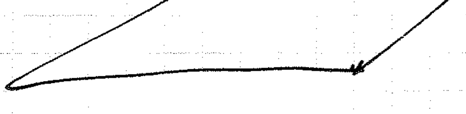
a projective curve differs from an affine curve by adding points ~~at~~ on the line at ∞ ~~correspon~~ corresponding to asymptotic values of the curve.

Example $P(X,Y) = X^3 - XY^2 + 10X^2 + 20Y + 30$.





$Q(X,Y,Z) = X^3 - XY^2 + 10X^2Z + 20YZ^2 + 30Z^3$



~~What does Q(X,Y,~~

$\{Q(X,Y,1) = 0\} = \{P(X,Y) = 0\}$.

What does $\{Q(X, Y, \frac{1}{2}) = 0\}$ looks like?

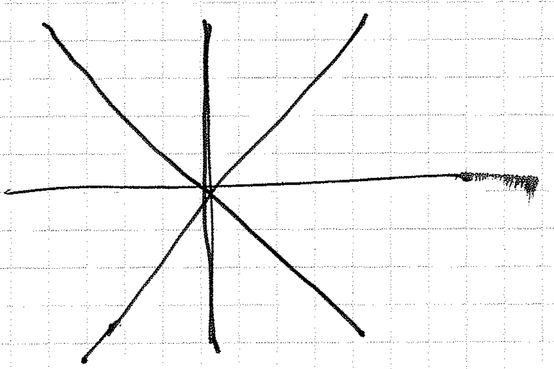
Looks like the same curve where we have stepped further back, rescaled the x and y coordinates.

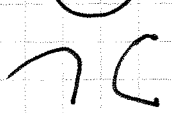
Terms of non-maximal degree are getting smaller.

$\{Q(X,Y,0) = 0\}$ consists of 3 lines through the origin corresponding to asymptotes of the original curve

$$Q(X,Y,0) = X^3 - XY^2 = \cancel{X(X^2} \; X(X-Y)(X+Y).$$

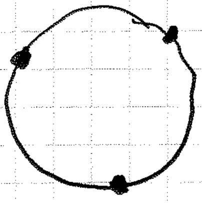
homogeneous polynomial in two variables.





affine curve

to the affine curve



projective curve corresponds to adding 3 points corresponding to the 3 asymptotic lines,

Topologically a circle.

~~Since the construction~~

Since this construction is purely algebraic we can perform it over any field ~~F~~ ~~In particular it~~ (For consistency we should call the locus a "curve".) makes sense over $\mathbb{C}$ though it is not so easy to draw.

A complex projective curve corresponds to a complex affine curve where we have added points corresponding to "asymptotic values". ~~or~~

Lemma. If $P(X,Y)$ is a non-linear homogeneous polynomial of degree d in two variables with $\mathbb{C}$ coefficients then it factors as a product of linear polynomials

$$P(X,Y) = \prod_{i=1}^{d} (\alpha_i X + \beta_i Y)$$

Proof. $$P(X,Y) = \sum_{\nu=0}^{d} a_\nu X^\nu Y^{d-\nu} = Y^d \sum_{\nu=0}^{d} a_\nu \left(\frac{X}{Y}\right)^\nu$$

Let e be the largest index with $a_e \neq 0$.

$$= Y^d \sum_{\nu=0}^{e} a_\nu \left(\frac{X}{Y}\right)^\nu$$

$$= a_e Y^d \prod_{j=1}^{e} \left(\frac{X}{Y} - \gamma_j\right)$$

$$= a_e Y^{d-e} \prod_{j=1}^{e} (X - \gamma_j Y).$$

(Classification of 0-dim projective varieties)

These lines correspond to points added to the projective curve.