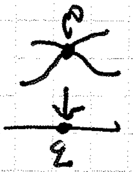
Lemma. Let f be a holomorphic function on an open nbd. U of 0 in $\mathbb{C}$ with $f(0)=0$ but f not identically 0. Then there is a unique integer $k \geq 1$ such that on some smaller nbd U' of 0 we can find a holomorphic function g with $g'(0) \neq 0$ and $f(z) = g(z)^k$ on U'.

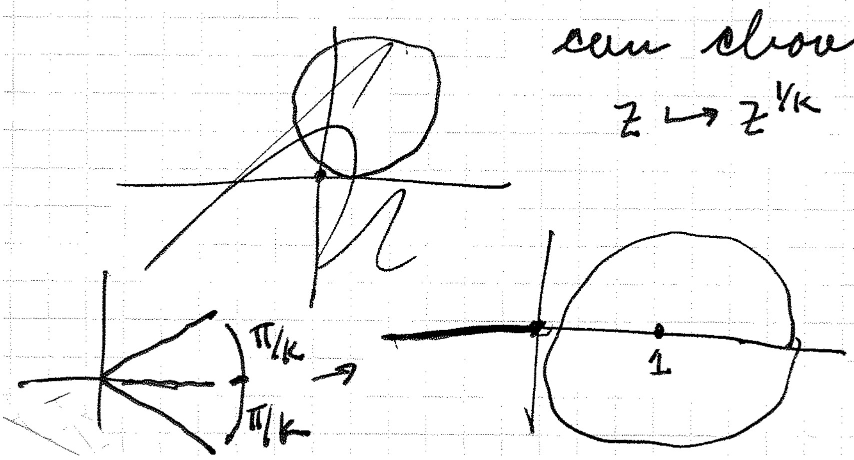


$k = \nu_f(p)$ is called the branching order of f at p. If $\nu_f(p) \geq 1$ p is called a ramification point and q is called a branch point.

Proof. $f(z) = a_k z^k + a_{k+1} z^{k+1} + \dots$ $\quad a_k \neq 0$

$$f(z) = a_k z^k (1 + b_1 z + b_2 z^2 + \dots) \qquad \text{where } b_j = \frac{a_{k+j}}{a_k}.$$

Assume $U' \subset U$ is small enough that $|\sum b_j z^j| < 1$ then the image of $z \mapsto (1 + b_1 z + b_2 z^2 + \dots)$ is contained in a disk in $\mathbb{C} - \{0\}$. In particular we can choose a branch of $z \mapsto z^{1/k}$ in this disk and define $h(z) = (1 + b_1 z + b_2 z^2 + \dots)^{1/k}$



⑨

Let $g(z) = a_k^{1/k} z\, h(z)$ for some choice of k-th root

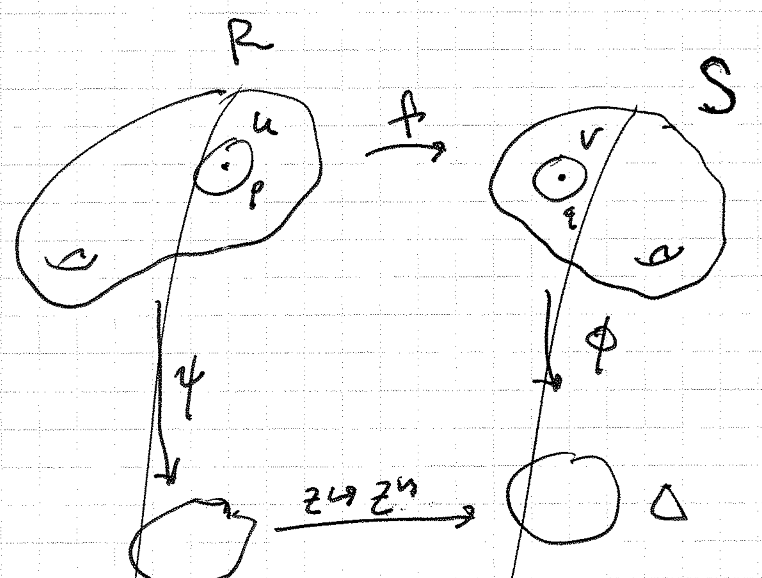
then $g^k(z) = a_k z^k (1 + b_1 z + b_2 z + \dots) = f(z)$.

Furthermore ~~g~~ since $g'(0) = a_k^{1/k} \neq 0$

we have that g is locally invertible with a holomorphic inverse

Theorem. (local model for holomorphic maps between Riemann surfaces)

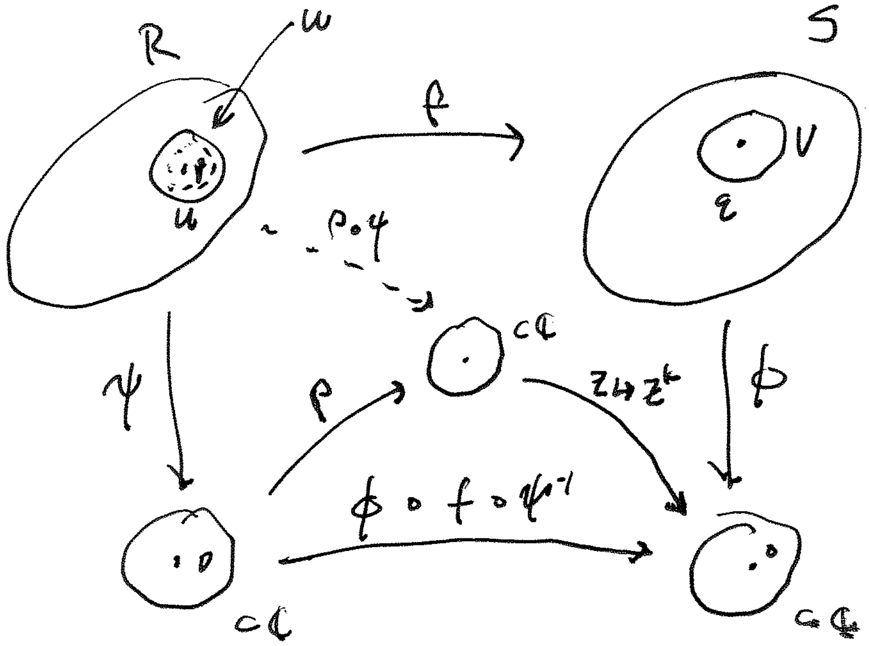
Let $f: R \to S$ be a holomorphic with $f(p) = q$ and f not constant near p.
Given any chart $\phi: V \to \Delta$ with $\phi(q) = 0$ there is a chart $\psi: U \to \Delta$ with $\psi(p) = 0$ and
$(\psi \circ f)(z) = (\phi(z))^n$.



(makes sense indep. of chart)

Proof. Let $g(z) = (\phi \circ f)^{1/n}$. ~~$g'(p)$~~ $g'(p) \neq 0$ so g is locally invertible on a smaller chart U'. Let $\psi = g|U'$. Thus ~~(ψ)~~ $\psi^n(z) = g^n(z)$
$= \phi \circ f$.

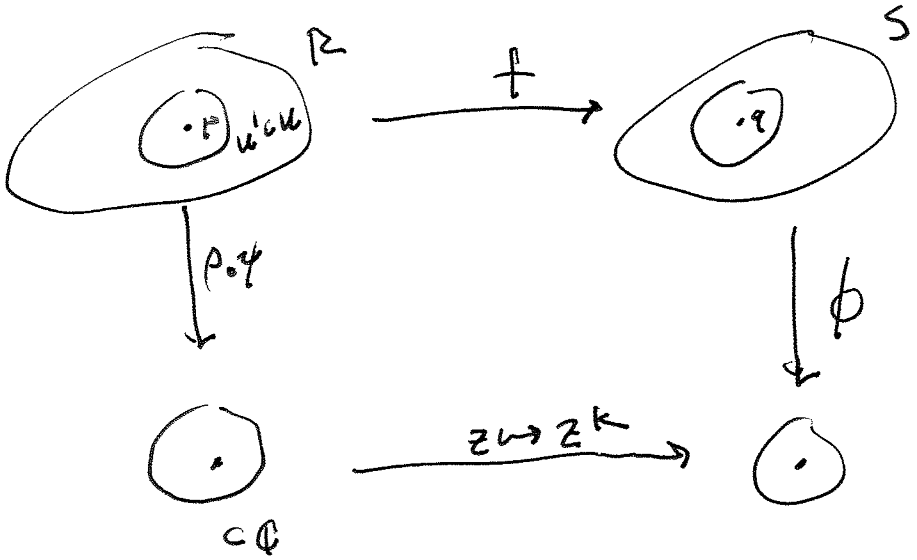
Geometric interpretation of the order of f: it is the n so that for q' near q $f(w) = q'$ has n solns for w near p.



Comps. are defined where they make sense.

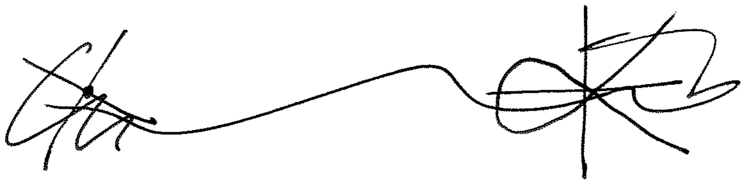
Let k be the order of the zero of $\phi\circ f\circ\psi^{-1}$.

Let $\rho = (\phi\circ f\circ \psi^{-1})^{1/k}$



We call $k = \nu_f(p)$ the branching order of f at p.

If $\nu_f(p) > 1$ then p is called a ramification point and q is called a branch point.



Note that we can geometrically characterize k as the # of solutions ~~for~~ of $f(z)=q_0^*$ for q_0^* near q and ~~z near~~ z near p since this is the behavior of $z \mapsto z^k$.

If f is locally injective (if $k=1$) then f is locally invertible.

Prop. A bijective holomorphic map from R to S has a holomorphic inverse.

Want to move from a local picture to a global picture.

~~Example of a loc~~

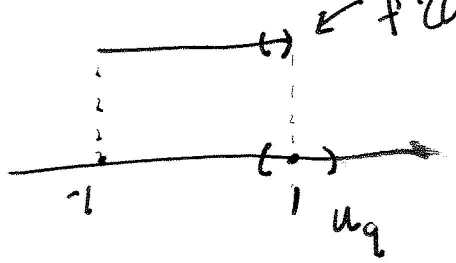
Local homeomorphisms and covering maps.

Def. $f: X \to Y$ is a local homeomorphism if for each $p \in X$ there is a nbd. $p \in U_p \subset X$ with $f|U_p$ a homeomorphism from U_p to $f(U_p)$.

Example: $f: R \to S$ $f' \neq 0$. No ramification points.

Def. $f: X \to Y$ is a covering map if for each $q \in Y$ there is a nbd U_q so that $f^{-1}(U_q)$ is a disjoint union of sets $\tilde{U}_{p_i}$ with $f: \tilde{U}_{p_i} \to U_q$ a homeomorphism.

Not every local homeomorphism is a covering map. Let Δ be the open unit disk then the inclusion $\iota: \Delta \to \mathbb{C}$ is a local homeomorphism but not a covering map. In particular the point 1 is not evenly covered



The advantage of covering maps is that there study reduces to ~~that~~ the analysis of the fundamental group of the base...
No such algebraic theory for local homeomorphisms.

$\boxed{\text{We can do path lifting for covering spaces.}}$

Proposition. Let $f: X \to Y$ be a local homeomorphism. If X is compact then f is a covering space of finite ~~degree~~ covering degree. (X, Y Hausdorff)

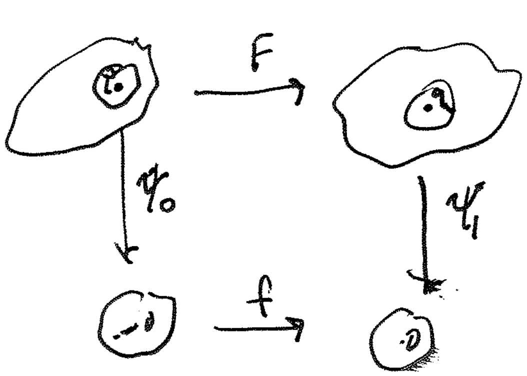
Proof. Let $q \in Y$. ~~$f^{-1}q$~~ If $f(p) = q$ then there is a nbd. U_p so that $f|U_p$ is 1-1 so the inverse images of q are isolated. In particular $f^{-1}(q)$ is finite. Let $p_1 \dots p_k$ be these points. Let $U_1 \dots U_k$ be nbds of $p_1 \dots p_k$ on which f is a homeo. Now $X - \bigcup U_j$ is ~~comp~~ closed so it is compact so $f(X - \bigcup U_j)$ is ~~a~~ compact. Let U_q be a nbd. of q disjoint from $f(X - \bigcup U_j)$.
So $f^{-1}(U_q) \subset \bigcup U_j$. Let $\hat{U}_j = U_j \cap f^{-1}(U_q)$.
~~f is a hom~~ $f|\hat{U}_j$ is a homeo. So U_q is evenly covered.

If ~~a holomorphic~~ a holomorphic ~~function~~ map f a is locally constant the multiplicity $v_f(p)$ does not make sense.

Prop. If $f: R \to S$ is holomorphic, and R is connected and f is locally constant at a point then f is constant.

Proof. Let $X = f^{-1}(q)$ for $q \in S$. X is closed since f is continuous.
If $f(p) = q$ but f is not locally constant at p then p is an isolated point of X.
(map looks like $z \mapsto z^n$ with $n \geq 1$ in a nbd. of p) where $p = q = 0$.
~~The union of x the set of isolated~~

Write X as $I \cup N$. Since $I \subset$ a open set U disjoint from N, $N = X \cap U^c$ is closed. On the other hand N is open. If $p \in N$



~~f is zero on~~ 0 is the limit of points where f is 0 so $f(z) = 0$ in a nbd. of 0 so $F(p) = q$ in a nbd. of p. If N is open, closed non-empty and R is connected then $N = R$ an f is constant

~~The set of zeros of $z \mapsto z^k$ is a point if k~~

Some results follow directly from the local picture of holomorphic maps. I want to mention these first and then return to the analysis of covering spaces etc.

Local model:

$$\begin{tikzcd}
\cdot p \arrow[r, "F"] \arrow[d, "\phi_p"] & \cdot q \arrow[d, "\phi_q"] \\
\cdot 0 \arrow[r, "f"] & \cdot 0
\end{tikzcd}$$

set up so that $f(0)=0$.

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 \ldots$$

If the power series does not vanish then ~~we~~ we can change coordinates ~~so~~

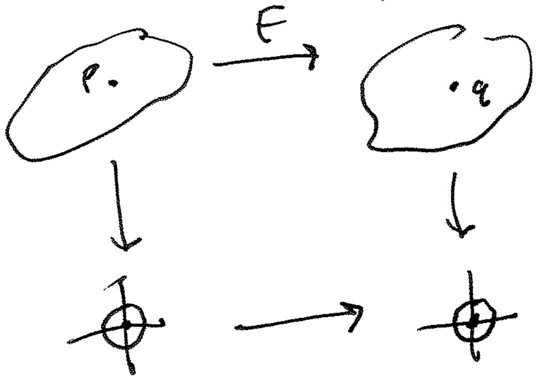
so $f(z) = z^k$ (a_k first non-zero coef.)

If the power series vanishes then

$f(z) \equiv 0$. ~~so~~ so $F(p) \equiv q$ in a nbd. of p.

Prop. ~~If $f:R\to S$ is not~~ If R is connected, $f:R\to S$ is holomorphic and not constant then the image of R is ~~constant~~ open.

Proof. ~~The map~~ This follows from the fact that $z\mapsto z^n$ ~~surjects onto a nbd. of 0 if $n\geq 1$.~~ takes any nbd. of 0 to a nbd. of 0.



Cor. If R is compact and connected, f is holomorphic and non-constant then ~~the~~ $f(R)$ is a component of S.

Cor. ~~An affine curve~~ A holomorphic function on a compact Riemann surface is constant.

Proof. ~~Then~~ $f:R\to \mathbb{C}$. If f is non-constant then ~~the~~ image is a component of $\mathbb{C}$ so it is all of $\mathbb{C}$. But $\mathbb{C}$ is not compact

Definition. A function $f: X \to Y$ is proper if the inverse image of a compact set is compact.

Proposition. Let $f: X \to Y$ be a local homeomorphism, Y be locally compact and f proper. Then f is a covering map.

Proof. Given $q \in Y$ we can find a nbd. U_q with $\bar{U}_q$ compact. Consider the restriction of f to $f^{-1}(\bar{U}_q)$. This set is compact. Apply previous result.

Proposition. Let $f: R \to S$ be a holomorphic map of Riemann surfaces which is not locally constant at any point. Assume f is proper. For any $q \in S$ there is a nbd U_q so that $f^{-1}(U_q)$ is a disjoint union of sets U_{p_i} and $f|U_{p_i}$ is conjugate to $z \mapsto z^k$ where ~~k is the~~ $k = V(p_i, f)$.

~~A proper map~~

Cor. If $f : R \to S$ is holomorphic, ~~and~~ not locally constant, ~~and S is connected~~ then the quantity ~~Σ_4~~ $q \mapsto \sum_{p_j \in f^{-1}(q)} v_{p_j}(f)$

is locally constant.

Proof. ~~For~~ the map $f : z \mapsto z^k$ this quantity is locally constant.

(at $q = 0$ there is 1 inverse image with $v_0(f) = k$. For $q \neq 0$ but q small there are k inverse images with $v(f) = 1$ at each.)

Def. If S is connected then this # is constant and we call this number the degree of the map. $d(f)$.