Circle 2015
Homeomorphisms

The support class for this module meets at 3pm on Mondays. This will start one week late on Oct. 19 instead of Oct. 12. notes available on my personal website under MA424.

Dynamical systems deals with the long term behavior of simple systems.

We are interested in systems which are deterministic, finite dimensional and autonomous.

Deterministic: Current state of the system determines the state of the system at all times

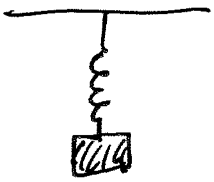
Finite dimensional: The state of the system is determined by finitely many parameters.

Autonomous: The rules governing the system do not change with time.

The systems we consider can come from a variety of fields. physics, biology, engineering or maths. If you have taken MA254 or MA371 you have seen examples.

One source of systems is ODE. Here is a simple example:

Harmonic oscilator



The system is modelled by the ODE $\ddot{x}(t) = -kx(t)$ where x is the deviation from rest position and k is related to the weight of the mass of the object and the strength of the spring.

The system is deterministic in that knowing the position and velocity now determines position and velocity for all time. Rewrite our equation as a first order ODE in these variables

$$\dot{x}(t) = y(t)$$
$$\dot{y}(t) = -k\cancel{X}(t). \quad = \cancel{X}$$

We call $\mathbb{R}^2 = \{(x, y)\}$ phase space.

Let $\phi_t(x_0, y_0)$ be the solution with $\phi_0(x_0, y_0) = (x_0, y_0)$.

The orbit of $p = (x_0, y_0)$ is the set $\{\phi_t(x_0, y_0) : t \in \mathbb{R}\}$.

In this case the orbits are ellipses that fill up $\mathbb{R}^2$.

In this case the orbits are also periodic

$\cancel{\phi_t(x_0,y_0) =}$ $\phi_\omega(x_0, y_0) = (x_0, y_0)$ where $\omega = \frac{2\pi}{\sqrt{k}}$.

If we had a friction term  then orbits would be spirals. Not periodic

~~Historically such systems~~

The fact that the system is autonomous is reflected in the fact that the coefficient functions do not depend on time.

Given an autonomous system of ODE's there are two methods of approaching it. ~~Often~~

We can start with $p \in X$ and analyze $\phi_t(p)$ as t changes. This is particularly useful when we have an explicit solution.

Alternatively we can fix t and think of $p \mapsto \phi_t(p)$ as a map from $X \to X$.

"$\mathbb{R}^n$"

This shift in perspective allows us to bring to bear different tools and allows us to address ~~and different~~ new questions.

Write $f^t : X \to X$ for the map ~~ϕ~~ $f^t(p) = \phi_t(p)$.

Thm. Consider an autonomous first order ODE in n variables with C^k coefficients so that solutions are defined for all time then

⓪ $f^s : X \to X$ is C^k

① $f^{s+t} = f^s \circ f^t$

② $f^0 = Id_X$.

Cor. f^s is a C^k diffeomorphism with $(f^s)^{-1} = f^{-s}$.

Proof $f^s \circ f^{-s} = f^{s-s} = f^0 = Id$

$f^{-s} \circ f^s = f^{-s+s} = f^0 = Id.$

Def. $\{f^s, X\}$ satisfying ① and ② is called a flow or an action of $\mathbb{R}$ on X.

Example: The logistic equation.

Say that $x_n \in \mathbb{R}$ represents the population of an insect species after n generations. ~~at~~

Say that $x_{n+1} = F(x_n)$ where $F(x) = \underset{\text{growth}}{a}x - \overset{\text{limited resour}}{b}x^2$.

Then the population after n years is $F^n(x_0)$

Autonomous here means we use the same F each year. We have $x_n = F^n(x_0)$. If $X = \mathbb{R}$ (or $\mathbb{R}_+$) we get an action of the semi-group $\mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 0\}$ on X

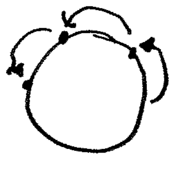
ie a map $f : \mathbb{Z}^+ \times X \to X$ satisfying (1) and (2).

Example: Rotation of the circle.

Let $X = \{z \in \mathbb{C} : |z| = 1\}$. Say $\lambda \in \mathbb{C}$ satisfies $|\lambda| = 1$.

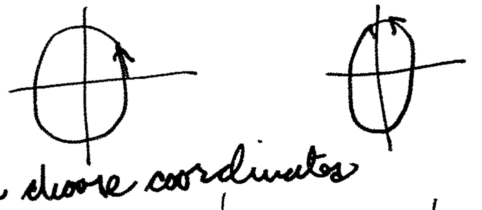
Let $M_\lambda : X \to X$ be defined by $M_\lambda(z) = \lambda z$.

$f : \mathbb{Z} \times X \to X$ given by $f^n(z) = M_\lambda^n$ defines an action of $\underline{\mathbb{Z} \text{ on } X}$,



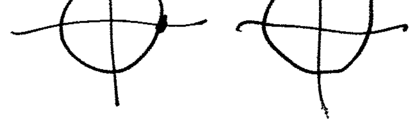
This example arises in number theory, questions of rational approximation of irrational numbers.

This last example is closely related to a system of 2 harmonic oscilators with $k_1 \neq k_2$.



Want to know how closely correlated the two oscilators are.

Can choose coordinates so that we have 2 circles.



Move around them at different rates. Set up a strobe light which goes off whenever the first oscilator returns to 0.

~~The effect on the second circle after~~
After ~~this~~ we rotate once around the first circle the "phase" of the second oscilator multiplies by a fixed z. After n revolutions the position of the second oscilator corresponds to M_z^n(initial position). (We will return to this later)

In this course dynamical systems will mean semi-group actions of G on X where G is $\mathbb{Z}$, $\mathbb{Z}_+$ or $\mathbb{R}$. We will often exploit differentiability properties. ~~To begin with X will be a circle~~
In many cases X is a circle S^1 or a torus $S^1 \times S^1$ or $\mathbb{R}$ or $\mathbb{C}$.

Rigid rotation of the circle.

It will be useful for us to write the circle additively.

~~Let $\mathbb{R}/\mathbb{Z}$~~ Define a map $\exp : \mathbb{R} \to \mathbb{C}$ by $\exp(r) = e^{2\pi i r}$. Two real numbers r, s map to the same point in the circle if $r - s \in \mathbb{Z}$. Then $s = r+n$ so $e^{2\pi i(r+n)} = e^{2\pi i r} \cdot e^{2\pi i n} = e^{2\pi i r} = e^s$.

Write $\mathbb{R}/\mathbb{Z}$ for $\mathbb{R}$ with the equivalence relation $r \sim s$ if $r - s \in \mathbb{Z}$. Every r is equivalent to some pt in $[0,1]$. o~

Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be defined by $R_\alpha(x) = x+\alpha \bmod \mathbb{Z}$
$= x+\alpha \bmod 1$

$\mathbb{R}/\mathbb{Z}$ is a metric space in a natural way.

R_α is equivalent to M_z where $z = \exp(\alpha)$.

$R_\alpha^n(x)$
$= x+n\alpha$
mod

~~$\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$~~

~~$S^1 \to S^1$~~

Definition, If $f: X \to X$ is a discrete dyn. system then $p \in X$ is a periodic point if $f^n(p) = p$ for some $n > 0$.

Proposition For R_α: ~~S^1~~ $\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ the following are equivalent:

(1) α is rational

(2) R_α has ~~[illegible]~~ at least one periodic point

(3) Every point in $\mathbb{R}/\mathbb{Z}$ is periodic.

Proof. (3)$\Rightarrow$(2) Clear.

(1)$\Rightarrow$(3) If $\alpha = m/n$ Then $R^n_\alpha(x) = R^n_{m/n}(x) = x + n \cdot \frac{m}{n} \bmod 1 = x$ so every pt. is periodic.

(2)$\Rightarrow$(1). If $R^n_\alpha(x) = x$ Then $n\alpha + x = x \mod \mathbb{Z}$ so $n\alpha = 0 \mod \mathbb{Z}$ so $n\alpha = m$ and $\alpha = m/n$.

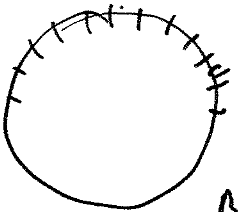
Proposition. If α is irrational then every orbit is dense in $\mathbb{R}/\mathbb{Z}$.

Proof. Assume α is irrational. Given $x \in \mathbb{R}/\mathbb{Z}$ and an open set $U \subset \mathbb{R}/\mathbb{Z}$ we need to find an n so that $R_\alpha^n(x) \in U$. Fix an $\varepsilon > 0$ so that U contains an ~~closed~~ interval I of length ε. Let $N > 1/\varepsilon$.

Consider the points $S = \{x, R_\alpha(x), R_\alpha^2(x) \dots R_\alpha^N(x)\}$.

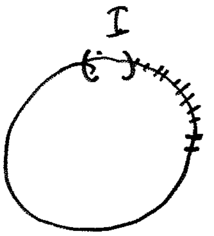
All of these points are distinct since if $R_\alpha^i(x) = R_\alpha^j(x)$ with $i < j$ then $x = R_\alpha^{j-i}(x)$ so x would be periodic and α would be rational.

Divide the circle into intervals of length $\frac{1}{N}$.



~~Then~~ By the pigeon hole principle one interval contains two distinct points in S. So $d(R_\alpha^i(x), R_\alpha^j(x)) \leq \varepsilon$.

But R_α preserves distances so $d(x, R_\alpha^{j-i}(x)) < \varepsilon$.



Now the iterates of $R_\alpha^{j-i}(x)$ are less than ε apart so one of them must be in I.

Thus $R^{m(j-i)}_{\cancel{\alpha m}} \left(R_\alpha^{j-i}\right)^m(x) \in I$ for some m.

$$\Vert$$

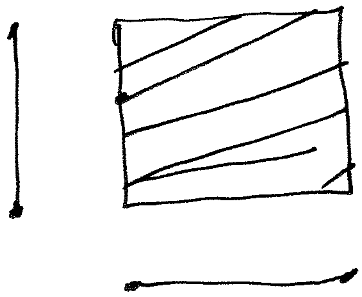
$R_\alpha^{(j-i)m}(x)$. Let $n = (j-i)m$

Conclusion. The nature of the number α completely determines the dynamics.

If we consider the family of maps R_α for $\alpha \in [0,1]$ then typically ~~if~~ α is irrational so typically every orbit of R_α is dense. since

If we think of a pair of harmonic osillators with distinct periods this means that typically there is no relation between the observed positions

Determinism versus "random" behavior.

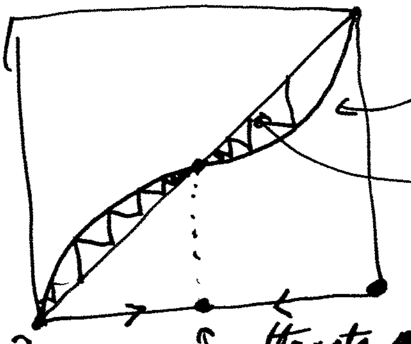


Our next topic is the dynamics of homeomorphis of the circle. Unlike ~~in~~ rotations homeomorphism need not preserve distance between pairs of points and this leads to some new phenomena.

If one point is periodic for a rotation then all points are periodic. For homeomorphisms this need not be the case.

Example: Let $f(x) = x + \frac{1}{10}\sin(2\pi x) \mod 1$

~~[illegible]~~
A periodic point p is a sink if there is a nbd^N of p to $q \in N$ $\Rightarrow d(f^n(p), f^n(q)) \to 0$ as $n \to \infty$,



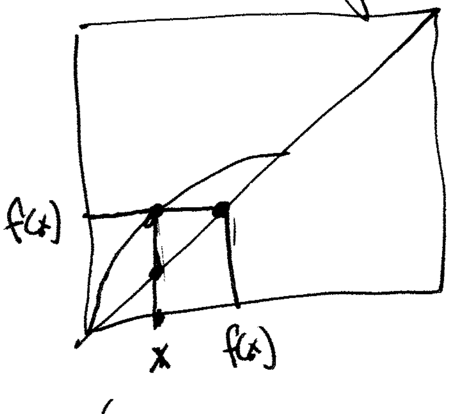
0 and $\frac{1}{2}$ are the only fixed point

repels points in forward time: source

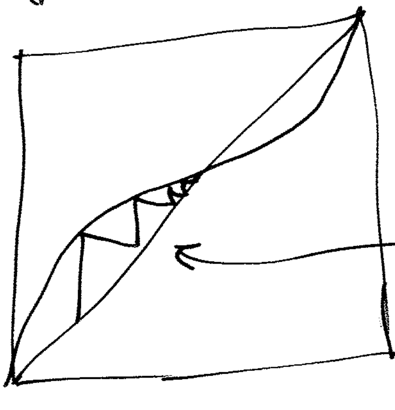
attracts pts. in forward time: sink

How do we iterate using the graph?

Graph gives a map from x variable to y variable. Diagonal gives a map from y variable back to x.



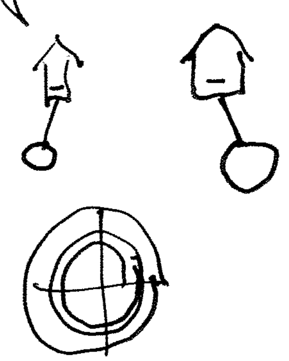
~~Go up or down to the graph then go~~ ~~Go vertically~~
Start on the diagon
Go vertically to the grap
Go horizontally to the diagonal.

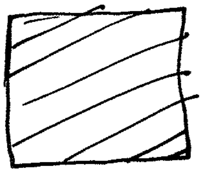


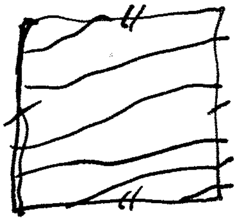
Think of the diagon as representing the phase space!

We discussed earlier how the dynamics of R_α related to a pair of harmonic oscilators. What about homeomorphisms? Homeomorphisms arise in the study of "weakly coupled" oscilators.

~~Say we have~~ Say we have two stable oscilators with some weak interaction. Different periods.





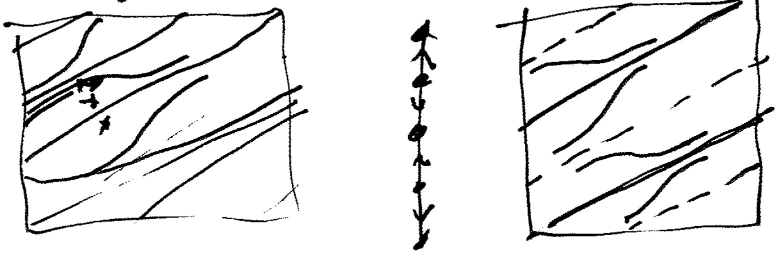


~~Ab~~ ~~A~~ Flow still corresponds to a flow on a torus. We can consider the first return map to the left hand edge. In this case it is a ~~diffeomo~~ diffeomorphism of the circle.

New phenomenon: ~~stable periodic~~ Periodic sinks.

 or . Periodic sinks give us orbits which approach periodic ~~pe~~ orbits in forward time.

"Phase locking".

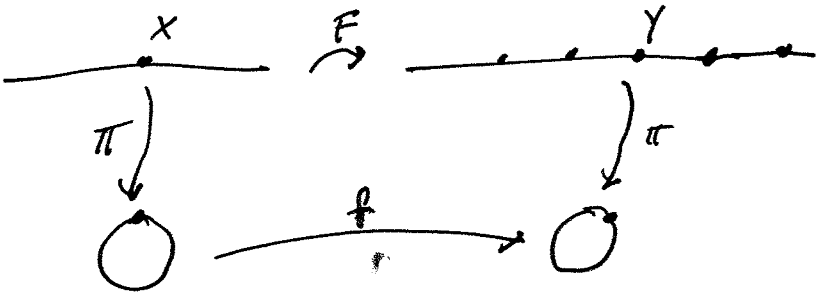


Periods of the two oscilators are rationally related.

This can be a stable condition in this setting as opposed to the case of the uncoupled oscilators.

Recall that a lift $F:\mathbb{R}\to\mathbb{R}$ of a map $f:\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ is a map such that $f(\pi(x))=\pi(F(x))$.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathbb{R} \arrow[d, "\pi"] \\
\mathbb{R}/\mathbb{Z} \arrow[r, "f"] & \mathbb{R}/\mathbb{Z}
\end{tikzcd}$$

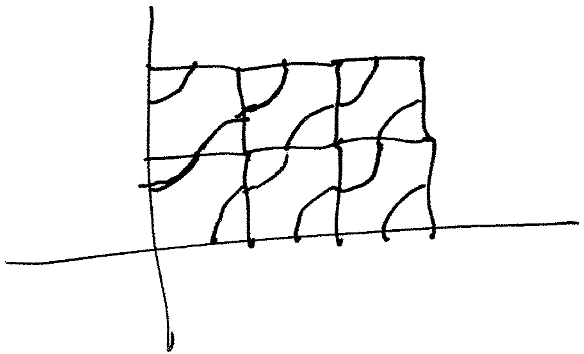


Given $x\in\mathbb{R}$, ~~$F(x)=Y$ satisfies the equation~~ the possible values for $Y=F(x)$ differ from one another by integers.

Thm. Lifts of f exist.
Proof depends on the observation that if we have a particular x, Y so that $f\pi(x)=\pi(Y)$ then we can construct a lift of f in a nbd. of x. We build the lift by piecing together these local ~~lf~~ lifts

Interesting to picture, for each x the set of all possible Y's so that $f\pi(x)=\pi(Y)$: $\{(x,Y): f\pi(x)=\pi(Y)\}$

a lift ~~is a~~ of f is a subset of this picture which is a graph of a continuous functio



The lift of f is built locally but captures some global information:

Proposition. ① If F and G are lifts of f then $G(x) = F(x) + k$ for some fixed $k \in \mathbb{Z}$.

② ~~F(x) F(1~~ $F(x+1) = F(x) + d$ for some fixed $d \in \mathbb{Z}$.

Proof. ① $F(x) - G(x)$ satisfies $\pi(F(x) - G(x)) = \pi(F(x)) - \pi(G(x))$ $= f(x) - f(x)$ ~~mod Z~~ $= 0$ mod $\mathbb{Z}$ so $F(x) - G(x) \in \mathbb{Z}$ but $F(x) - G(x)$ is a continuous function so it must be constant

② $\pi(F(x+1) - F(x)) = \pi(F(x+1)) - \pi(F(x)) = f(x) - f(x) = 0 \bmod \mathbb{Z}$.

As before it must be constant

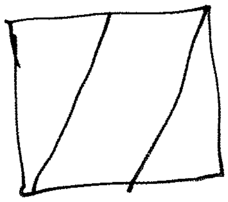
~~Prop.~~ Def. The degree of $f : \mathbb{R}/\mathbb{Z} \circlearrowleft$ is $F(x+1) - F(x)$.

By ① this is independent of the lift F since $(G(x+1) - G(x)) - (F(x+1) - F(x)) = (G(x+1) - F(x+1))$ $- (G(x) - F(x)) = k - k = 0$.

By ② this number is independent of x.

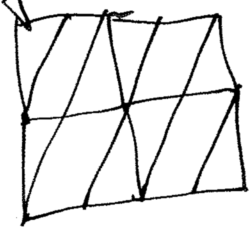
Example①. $f(x)=2x \mod 1$.

Note that this is well defined since $f(x+k) = 2(x+k) = 2x+2k$
$= 2x \mod 1$.

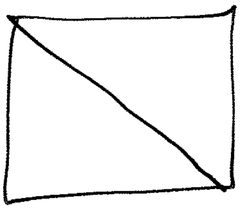


$F(x) = 2x$ is a lift of f.

$\deg f = F(1) - F(0) = 2\cdot 1 - 2\cdot 0 = 2$.



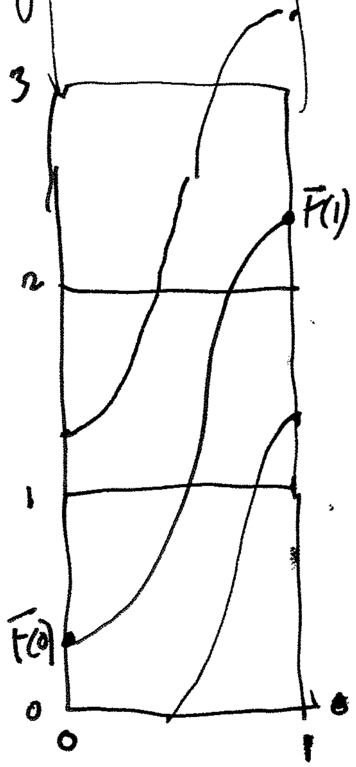
② $f(x) = -x \mod 1$



Example 2 is a homeomorphism while example 1 is not.

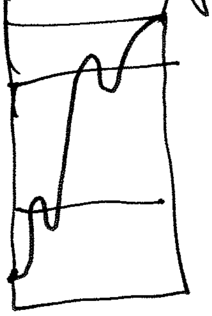
Prop. If f is a homeomorphism then $\deg f = \pm 1$.

Proof.



Degree 2. $f^{-1}(0)$ has 2 (or more) points so f is not injective.

~~Same argument shows~~



$\overset{x}{\overset{\shortparallel}{f^{-1}(1)}}$ and $\overset{x'}{\overset{\shortparallel}{f^{-1}(2)}}$

both satisfy

~~$f(x) = f(x') \mod \mathbb{Z}$~~

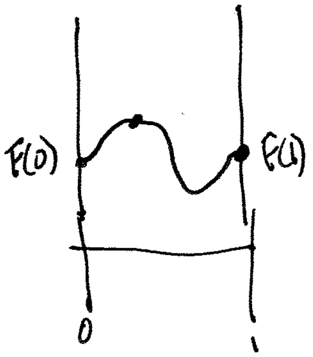
~~$f(x) = f(x') = F(x) = F(x')$~~

$f(x) = F(x) \mod 1 = 1 \mod 1 = 0$

$f(x') = F(x') \mod 1 = 2 \mod 1$

Same argument shows $|\deg f| \leq 1$ so $\deg f = -1, 0, 1$.

$\deg f = 0$ ~~violates the mean va~~ cannot be injective due to the mean value thm.



~~The case of degree 1 corresponds to orien~~

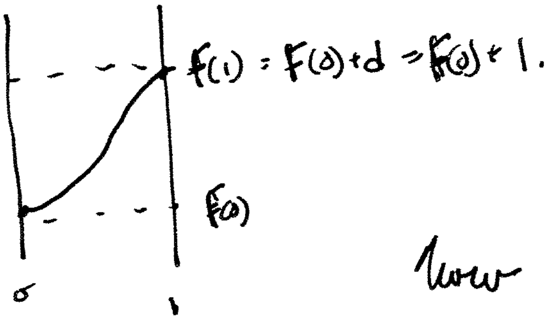
What is different about $\deg f = 1$ and $\deg f = -1$?

~~There is~~ In terms of the lift F $\deg f = 1$ corresponds to a monotone increasing function, $\deg f = -1$ corresponds monotone decreasing

~~This can be stated in terms of orientation of the circle:~~

Prop. If f is a homeomorphism with degree 1 then F is monotone increasing.

Proof.



$F([0,1]) = [F(0), F(1)]$

[illegible]

now [illegible] $F([n, n+1])$

now [illegible]

$= [F(n), F(n+1)] = [F(0)+n, F(1)+n]$

using $F(x+1) = F(x)+1$ repeatedly.

Lemma. Let F be a monotone increasing function from $\mathbb{R}$ to $\mathbb{R}$ (for example a lift of a degree 1 circle homeomorphism). If $F(x) \geq x$ then $F^k(x) \geq x$ for all $k > 0$. If $F(x) \leq x$ then $F^k(x) \leq x$ for $k \geq 0$.

Assume $F(x) \geq x$

Proof. We prove that $x \leq F(x) \leq F^2(x) \leq \dots \leq F^k(x)$ (*) by induction on k. Statement is true for $k=1$ by assumption. Assume the statement for k so $x \leq F(x) \leq \dots F^k(x)$. By monotonicity we have $F(x) \leq F^2(x) \leq \dots F^k(x) \leq F^{k+1}(x)$. Combine this with $x \leq F(x)$ and we have our statement for $k+1$.

Our next objective is to show that $\lim_{n\to\infty} \frac{F^n(x)}{n}$ exists. We will use repeatedly the fact that $F(x+1) = F(x)+1$. We can rewrite this. Let $T(x) = x+1$. Then we have $F\circ T = T\circ F$ since $F\circ T(x) = F(x+1)$ and $T\circ F(x) = F(x)$

(This can be cast in terms of a $\mathbb{Z}^2$ action on $\mathbb{R}$).

Example: $(T^n\circ F^m)^p = (\underbrace{T^n\circ F^m\circ T^n\circ F^m\circ \quad T^n\circ F^m}_{p})$

$$= T^{pn}\circ F^{pm}.$$

We would like to capture the behavior of f with a number that plays a role like the α in R_α. α is the increase in the x coordinate of a lift of R_α. In general the ~~in~~ increase in the x coordinate of a lift F of f is not constant so we consider the average increase along the orbit.

Let $f : \mathbb{R}/\mathbb{Z} \circlearrowleft$ and F be a lift

Def. $\rho(F) = \lim_{n \to \infty} \frac{F^n(x)}{n}$

and $\rho(f) = \rho(F) \bmod 1$.

We call $\rho(f)$ the rotation number of f.

In order to make this well defined we need to show that $\rho(F)$ is independent of x and $\rho(f)$ is independent of the choice of the lift.

Example. ~~$\rho(R_\alpha) = \alpha$~~ $f = R_\alpha$, $F = T_{\alpha+k}$ where

$T_{\alpha+k}(x) = x + \alpha + k$. Then $\rho(T_{\alpha+k}) = \lim_{n \to \infty} \frac{n(\alpha+k)+x}{n} = \alpha +$

$\rho(R_\alpha) = \alpha + k \bmod 1$.

Lemma. $\lim_{n\to\infty} \frac{F^n(0)}{n}$ exists.

Proof. Say that $F^n(0) \in [k, k+1]$ for some $k \in \mathbb{Z}$

then $F^n(0) \geq k$ so $T^{-k}F^n(0) \geq 0$.

$$(\underbrace{T^{-k}F^n}_{\text{monotone increasing}})^m(0) \geq 0 \text{ for all } m.$$

Do we really need to show that $T^{-k}F^n$ is a lift? suffices to know $T^{-k}F^n$ is monotone increasing.

Thus $T^{-km}F^{nm}(0) \geq 0$

so $F^{nm}(0) \geq T^{km}(0) = km$ for all m

· ~~lim sup $F^{nm}(0)$~~

Similarly iff $F^n(0) \leq k+1$ so $F^{nm}(0) \leq (k+1)m$

$\downarrow$

$$T^{-k-1}F^n(0) \leq 0 \longrightarrow (T^{-k-1}F^n)^m(0) \leq 0 \longrightarrow T^{-m(k+1)}F^{nm}(0)$$

Thus $km \leq F^{nm}(0) \leq (k+1)m$

so

$$\frac{km}{mn} \leq \frac{F^{nm}(0)}{mn} \leq \frac{(k+1)m}{mn}$$

$$\frac{k}{n} \leq \frac{F^{nm}(0)}{mn} \leq \frac{(k+1)}{n}$$

$$\frac{F^{nm}(0)}{mn} \in \left[\frac{k}{n}, \frac{k+1}{n}\right].$$

Pretty good estimate but only for a subsequence.

True for all $\underline{m}$.

$$\left| \frac{F^n(0)}{n} - \frac{F^{nm}(0)}{nm} \right| \le \frac{1}{n}$$

So for any $n, m \in \mathbb{N}$ we get

$$\left|\frac{F^n(0)}{n} - \frac{F^m(0)}{m}\right| \le \left|\frac{F^n(0)}{n} - \frac{F^{nm}(0)}{nm}\right| + \left|\frac{F^m(0)}{n} - \frac{F^{nm}(0)}{nm}\right| \le \frac{1}{n} + \frac{1}{m}.$$

↖ add and substract $\frac{F^{nm}(0)}{nm}$

This shows that $\frac{F^n(0)}{n}$ is a Cauchy sequence:

Take $\varepsilon > 0$. For $n, m > \frac{2}{\varepsilon}$ we get

$$\left|\frac{F^n(0)}{n} - \frac{F^m(0)}{m}\right| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

In particular $\frac{F^n(0)}{n}$ converges.

~~Remark: Could have started with x instead of 0.~~

~~Get if $F^n(x) \in [x+k, x+k+1]$ then $\frac{F^{nm}(x)}{nm} \in$~~

Scholium: If $F^n(0) \ge k$ then $\rho(F) \ge \frac{k}{n}$.

~~If $F^n(x) - x \ge k$ then $\rho(F) \ge \frac{k}{n}$~~

~~If $F^n(0) \le k$ then $\rho(F) \le \frac{k}{n}$.~~ Uses convergence.

Uses $\rho(F) = \lim_{m \to \infty} \frac{F^{nm}(0)}{nm} \ge \frac{k}{n}$.

Also $F^n(x) - x \ge k$ then $\rho(F) \ge \frac{k}{n}$.

Lemma. $\lim_{n\to\infty} \frac{F^n(x)}{n}$ exists for all x and takes the same value.

Proof. For some $m\in\mathbb{N}$ $\quad -m\le x\le m$.

Thus $\quad F^n(-m)\le F^n(x)\le F^n(m)$ since F^n is monotone increasing

$$F^nT^{-m}(0)\le F^n(x)\le F^nT^m(0)$$

$$T^{-m}F^n(0)\le F^n(x)\le T^mF^n(0)$$

$$F^n(0)-m\le F^n(x)\le F^n(0)+m$$

$$\frac{F^n(0)-m}{n}\le \frac{F^n(x)}{n}\le \frac{F^n(0)+m}{n}.$$

So $\lim_{n\to\infty}\frac{F^n(x)}{n}=\lim_{n\to\infty}\frac{F^n(0)}{n}$.

Lemma. Let F and G be lifts of f Then

$\rho(F)=\rho(G) \bmod 1$.

Proof. $F(x)=G(x)+k$ for some k.

$$\lim_{n\to\infty}\frac{F^n(x)}{n}=\lim_{n\to\infty}\frac{(G(x)+k)^n}{n}=\lim_{n\to\infty}\frac{(T^kG)^n(x)}{n}$$

$$=\lim_{n\to\infty}\frac{T^{kn}G^n(x)}{n}$$

$$=\lim_{n\to\infty}\frac{G^n(x)+kn}{n}$$

$$=\lim_{n\to\infty}\frac{G^n(x)}{n}+k.$$

Now the rotation number is defined for homeomorph
deg 1

Our objective is to connect properties of the rotation number $\rho(f)$ to dynamical properties of f.

Let's look at rotation numbers.

Proposition. Let f be an orientation preserving homeomorphism of the circle then $\rho(f)=0$ if and only if f has a fixed point.

Proof. Assume $p=f(p)$. Let F be a lift of f.
~~Since~~ $F(p) \bmod \mathbb{Z} = f(p) = p$ so $F(p)-p \in \mathbb{Z}$. Say $F(p)-p=k$. Then ~~F(x) is~~ $G(x)=F(x)-k$ is also a lift of f and $G(p)=F(p)-k=p$.
Now $\rho(f) \equiv \rho(G) = \lim_{n\to\infty} \frac{G^n(p)}{n} = \lim_{n\to\infty} \frac{p}{n} = 0$. So $\rho(f)=$

~~Now suppo~~
Now we prove the converse. Assume that $\rho(f)=0$. Let F be a lift. We have $\rho(F)=k$. Let $G(x)=F(x)-k$ so $\rho(G)=0$.

Now $G(0)=0$ or $G(0)<0$ or $G(0)>0$. If $G(0)=0$ we are done. Assume $G(0)>0$. Then ~~G(0)~~ $G^n(0)$ is an increase sequence. Say $G^n(0)>1$ for some n. Then

according to our scholium we have $\rho(G) \geq \frac{1}{n}$.
This would contradict our assumption.
So $G^n(0)$ is increasing and bounded. Let
$\lim_{n\to\infty} G^n(0) = x^*$. Then ~~$\lim_{n\to\infty} G^*(G^n(0)) = G(x^*)$~~

$$G(x^*) = \lim_{n\to\infty} G(G^n(0)) = \lim_{n\to\infty} G^{n+1}(0) = x^*.$$

So x^* is fixed by G and $\pi(x^*)$ is fixed by f.

Prop. $\rho(f^n) = n\,\rho(f)$.

Proof. Let F be a lift of f. Then F^n is a lift of f^n.

~~$\rho(f^n) = \lim$~~

$$\rho(F^n) = \lim_{m\to\infty} \frac{F^{nm}(x)}{m} = n \lim_{n\to\infty} \frac{F^{nm}(x)}{nm} = n\cdot\rho(F).$$

Reducing mod 1 gives:

$$\rho(f^n) = n\,\rho(f).$$

True for $n<0$. Follows from $\rho(f^{-1}) = -\rho(f)$. Exercise.

Theorem. f has a periodic ~~point~~ orbit if and only f has a rational rotation number.

Proof. If f has a periodic orbit of period n then $f^n(p) = p$ so $\rho(f^n) \equiv 0 \bmod 1$. If $\rho(f) = \frac{m}{n}$ then $\rho(f^n) = n\cdot\frac{m}{n} = \cancel{n}\, m = 0 \bmod 1$ so f^n has a fixed point.

For future use we want to extract a little more from our lemma on rotation numbers.

Lemma. If f is an orientation preserving homeomorphism of $\mathbb{R}/\mathbb{Z}$ and F is its lift then if $F^n(x) \geq x+k$ we have $n\rho(F) \geq k$ for $n \in \mathbb{Z}$.

Proof. Assume $F^n(x) \geq x+k$

so $$F^n(x) \geq T^k(x)$$

$$T^{-k}F^n(x) \geq x$$

Using previous lemma:

$$(T^{-k}F^n)^m(x) \geq x \qquad \text{for any } m>0$$

$$F^{nm}(x) \geq x+km$$

$$\frac{F^{nm}(x)}{m} \geq \frac{x+km}{m}$$

Let $m \to \infty$ $$\rho(F^n) \geq k$$

so $$n\rho(F) \geq k.$$

~~A~~ A natural notion of equivalence in dynamical systems.

We say that $f:X\circlearrowleft$ and $g:Y\circlearrowleft$ are topologically conjugate if there is a homeomorphism $h:X\to Y$ so that $g\circ h = h\circ f$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

Why is this a good notion?

Behaviours we are interested in are topological: x has a dense orbit, x is periodic, x is attracted to a periodic orbit.

If a point ~~$x\in X$~~ $p\in X$ has a given ~~behav~~ topological behaviour in X (it is dense say) then $q=h(p)$ has the same topological behaviour in Y.

h takes orbits to orbits: $h(f^n(p)) =$

$hff\ldots f(p) = ghf\ldots f(p) = g\cdots g\,h(p) = g^n(q)$.

Since h is a homeomorphism it takes dense sets to dense sets.

Fits in with the ~~philosophy~~ idea of answering not how does $g^n(x)$ ~~look~~ $f^n(p)$ behave for a given p but rather given some behavior ~~for~~ is there some point p that realizes it??

Can also define a semi-conjugacy where we do not assume h is invertible. gives partial information. Not an equiv. relation.

~~Proposition.~~

~~The rotation number~~

~~A quantity attached~~ A quantity attached to a ~~dyn~~ dynamical system is a dynamical invariant if it takes the same value on top. conjugate systems. The rotation number is almost a topological invariant.

Proposition. Let circle homeomorphisms be topologically conjugate via a homeo. h. Then if $\deg h = 1$ $\rho_f = \rho_g$. If $\deg h = -1$, $\rho_f = -\rho_g$.

Proof. If $g \circ h = h \circ f$ then $g = h \circ f \circ h^{-1}$. Let H be a ~~f~~ lift of H and F a lift of f then $G = H \circ F \circ H^{-1}$ is a lift of g. Let $x = H(0)$

$$\cancel{\frac{G^n(x)}{n}} \qquad \frac{G^n(x)}{n} = \frac{H \circ F^n \circ H^{-1}(x)}{n} = \frac{H \circ F^n(0)}{n}$$

Say $F^n(0) \in [k, k+1]$ then $T^{-k} F^n(0) \in [0,1]$

$\boxed{H(x+1) = H(x)+d}$ $HT = T^d H$ where ~~t~~ d depends on deg H. $d = \pm 1$.

$$\frac{H\circ F^n(0)}{n} = ~~H(T^{-k}F^n(0)~~ \quad \frac{H(T^kT^{-k}F^n(0))}{n} = \frac{T^{\pm k}HT^kF^n(0)}{n}$$

Now if $T^{-k}F^n(0) \in [0,1]$ then $HT^kF^n(0) \in H([0,1])$

$$= \frac{HT^{-k}F^n(0) \pm k_n}{n} \in$$

$$\rho(G) = \lim_{n\to\infty} \frac{G^n(x)}{n} = \pm \lim_{n\to\infty} \frac{k_n}{n} = \lim_{n\to\infty} \frac{F^n(0)}{n} = \pm \rho(F).$$

Cor. Two rotations R_α, R_β are top. conjugate iff $\alpha = \pm\beta$.

Proof. Any map is top. conj. to itself. If $\alpha = -\beta$ then let $h(x) = -x$ mod 1 so ~~R_α~~ $h^{-1}R_\beta h(x) = -(\beta + -x) = x - \beta = R_\alpha(x)$.

On the other hand if R_α and R_β are top. conjugate then $\rho(R_\alpha) = \pm\rho(R_\beta)$ so $\alpha = \pm\beta$.

To what extent does rotation number capture ~~the~~ the idea of top. conjugacy?

~~If f has rational rotation num~~

If f has a periodic sink then f is not top equiv. to any R_α. Of course this can only happen if $\rho(f)$ is rational.

Last time we talked about the notion of topological conjugacy between dynamical systems.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

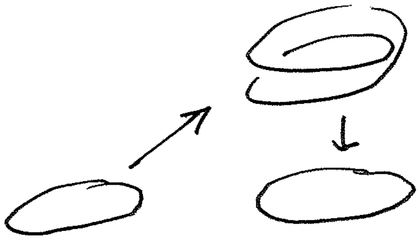
Sometimes useful to talk about a weaker notion of semi-conjugacy.

We say there is a semi-conjugacy ~~between~~ from (f,X) to (g,Y) if there is a continuous h that makes the diagram commute.

~~Here~~ In this case (g,Y) reflects some of the dynamics of (f,X) but perhaps not all since h can identify distinct points.

Example. There is a semi-conjugacy between $R_\alpha : \mathbb{R}/\mathbb{Z} \circlearrowleft$ and $R_{2\alpha} : \mathbb{R}/\mathbb{Z} \circlearrowleft$ given by $h(x) = 2x$ mod 1.

$$\begin{array}{ccc} h\circ R_\alpha(x) & = & R_{2\alpha}\circ h(x) \\ \parallel & & \parallel \\ 2(x+\alpha) & & 2x+2\alpha. \end{array}$$



In this case h is 2-1. Note that these maps are not ~~conju~~ top. conjugate unless $\alpha = \pm 2\alpha$ mod 1. Since the $\pm$ rotation number is a topological invariant.

②

Question: If f, g have the same irrational rotation number are they top. conjugate?

Theorem. If $\rho(f)=\alpha$ is irrational then ~~f is semi-conjugate to R_α~~. there is a semi-conjugacy from f to R_α.

~~Think of $x\in[0,1)$~~

Proof. Let $G(x)=x+\alpha$. ~~G is a lift of R_α. So $\rho(G)=\alpha$. Let F be a lift of f with $\rho(F)=\alpha$. (Recall that we can adjust $\rho(F)$ by an integer.~~

Let F be a lift of f with $\rho(F)=\alpha_0$ $(\in\mathbb{R})$ $\qquad \alpha_0\equiv\alpha$ m

Let $G(x)=x+\alpha_0$. So $\rho(F)=\rho(G)=\alpha_0$.

Pick $x_0\in\mathbb{R}$

Let $\Lambda_1=\{F^n(x_0)+m\,;\,m,n\in\mathbb{Z}\}$

$\Lambda_2=\{G^n(0)+m\,;\,m,n\in\mathbb{Z}\}$. $\qquad T^mF^n(x_0$

$\shortparallel$

Consider the maps ~~$L(m,n)$~~ $L_1:\mathbb{Z}^2\to\Lambda_1$, $L_1(m,n)=F^n(x_0)+m$

$L_2:\mathbb{Z}^2\to\Lambda_2$, $L_2(m,n)=G^n(0)+m$.

$\shortparallel$

Claim: L_1, L_2 are injective. ~~(Bijective with dense images.)~~ (The image of L_2 is dense. Follows from denseness of orbits of R_α). $\qquad T^mG^n$

If $L_1(m,n)=L_1(m',n')$ then $F^n(x_0)+m=F^{n'}(x_0)+m'$

so, reducing mod 1, $f^n(x_0)=f^{n'}(x_0)$ mod 1.

This gives $f^n(x_0)=f^{n'}(x_0)$ reducing mod 1 or

$f^{n-n'}(x_0)=x_0$. Since f has no periodic points

we have $n-n'=0$ or $n=n'$. ~~$T^mF^n(x_0)=T^{m'}F^n(x_0)$~~

$F^n(x_0)+m=F^n(x_0)+m' \Rightarrow m=m'$.

Lemma (with altered notation).

Let f be a homeomorphism with $\rho(f) = \rho$ irrational. Let F be a lift of f with $\rho(F) = \rho_0$.

Let $G(x) = x + \rho_0$ so G is a ~~f~~ lift of R_ρ. Pick $x_0 \in \mathbb{R}$.

Let $\Lambda_1 = \{ F^n(x_0) + m \,;\, m, n \in \mathbb{Z} \}$

$$\Lambda_2 = \{ \underset{\underset{n\rho_0 + m}{\parallel}}{G^n(0) + m} \,;\, m, n \in \mathbb{Z} \}$$

Define $H : \Lambda_1 \to \Lambda_2$ by $H(F^n(x_0) + m) = n\rho_0 + m$.

Then H is bijective, strictly increasing and $H(x+1) = H(x) + 1$

$HF(x) = GH(x)$. $(= H(x) + \rho_0)$.

Proof of Lemma. Consider the maps

$$L_1(m,n) = F^n(x_0)+m \overset{}{\;} \quad = T^m F^n(x_0) \qquad L_1 : \mathbb{Z}^2 \to \Lambda_1$$

$$L_2(m,n) = G^n(0)+m \quad = T^m G^n(0) \qquad L_2 : \mathbb{Z}^2 \to \Lambda_2 .$$

If $L_1(m,n) = L_1(m',n')$ then $F^n(x_0)+m = F^{n'}(x_0)+m'$ (*)

so, reducing mod 1, $f^n(x_0) = f^{n'}(x_0)$ mod 1.

This gives $f^{n-n'}(x_0) = x_0$ and $n-n'=0$ since f has no periodic points. Follows from (*) that $m=m'$!

Same argument shows L_2 is injective so $H = L_2 \circ L_1^{-1}$ is injective.

Now take $x_1, x_2 \in \Lambda_1$ with $x_1 < x_2$.

$x_1 = F^{n_1}(x_0) + m_1 < F^{n_2}(x_0) + m_2 = x_2$ for some n_1, n_2, m_1, m_2.

~~Let $y = F^{n_2}(x_0)$. Then $F^{-n_2}(y) = x_0$ xa~~

$$F^{n_1-n_2}(x_0) < x_0 + m_2 - m_1 \qquad \text{(recall } F, T \text{ commute)}$$

~~#~~ $\rho^{n_1-n_2} \le m_2 - m_1$ equality not possible.

~~If ρ were 0 then we could something about ρ and hence 0.~~

$$F^n(x_0)+m < F^{n'}(x_0)+m'$$

$$T^m F^n(x_0) < T^{m'} F^{n'}(x_0)$$

$$F^n(x_0) < T^{m'-m} F^{n'}(x_0)$$

$$F^n(x_0) < F^{n'} \circ T^{m'-m}(x_0)$$

$$F^{n-n'}(x_0) < T^{m'-m}(x_0)$$

$$F^{n-n'}(x_0) < x_0 + m'-m$$

Lemma on estimating rotation number from last class gives.

$$(n-n')\,\rho(F) \leq m'-m$$

Since $\rho(F)$ is irration
equality is not possible.

$$(n-n')\,\alpha_0 \leq m'-m$$

$$n\alpha_0 + m < n'\alpha_0 + m'$$

$$G^n(0)+m < G^{n'}(0)+m'$$

as was to be show

for $x \in \Lambda_1$

Claim. $H \circ F(x) = G \circ H(x)$

~~$H(F(F^n(x_0)+m)) = H(F(T^m F^n(x_0)))$~~

$$HF(x) = H(F(\underset{\overset{\shortparallel}{x}}{T^m F^n(x_0)})) = H(T^m F^{n+1}(x_0))$$

$$= T^m G^{n+1}(x_0)$$

$$= G(T^m G^n(x_0))$$

$$= GH(T^m F^n(x_0)) = GH(x)$$

~~$= GH(F^n(x_0)+m)$~~

for $x \in \Lambda_1$

Claim. $H(x+1) = H(x)+1$. Any $x = T^m F^n(x_0)$.

~~$H(T(F^n(x_0)+m)) = H(T^{m+1} F^n(x_0))$~~

$$H(T(T^m F^n(x_0))) = T^{m+1} G^n(x_0)$$

$$= T(T^m G^n(x_0))$$

$$= T(H(F^n(x_0)+m)).$$

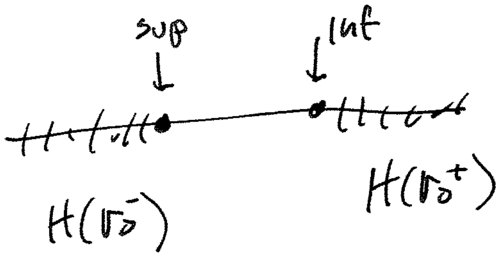
We want to extend H from $\Lambda_1 \to \Lambda_2$ to $\bar{H}: \mathbb{R} \to \mathbb{R}$. There is a unique way to do this preserving monotonicity (because Λ_2 is dense in $\mathbb{R}$

Let $r_0 \in \mathbb{R} - \Lambda_1$. Let $r_0^- = \{ r \in \Lambda_1 : r < r_0 \}$

$r_0^+ = \{ r \in \Lambda_1 : r > r_0 \}$.

Since H preserves order every element of $H(r_0^-)$ is less than every element of $H(r_0^+)$.

So $\sup H(r_0^-) \leq \inf H(r_0^+)$. Since $H(r_0^-) \cup H(r_0^+) = \Lambda_1$ and Λ_1 is dense in $\mathbb{R}$ we have $\sup H(r_0^-) = \inf H(r_0^+)$.

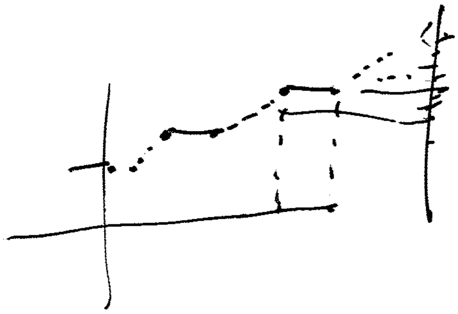


← picture of strict inequality.

Note that $H(r_0^-) \cup H(r_0^+)$ avoids an interval in $\mathbb{R}$. By

But $H(r_0^-) \cup H(r_0^+) = \Lambda_1$ is dense in $\mathbb{R}$ (by denseness of orbits of R_α).

A monotone map is continuous (the inverse



$\bar{H}$ takes any interval in ~~th~~ $\mathbb{R} - \Lambda_1$ to a point

$\bar{H}$ satisfies th $\bar{H} \circ T = T \circ \bar{H}$ so it induces a well defined map ~~[illegible]~~ $h(x): \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ by $h(x) = \bar{H}(x) \bmod 1$. ~~[illegible]~~ $h(x)$ ~~[illegible] $= H(x+1)$~~

~~$F \circ \bar{H} =$~~

$\bar{H} \circ F = G \circ \bar{H}$. So ~~h is a semi~~ $h \circ f = g \circ h$

and h is a semi-conjugacy.

This completes the construction of the semi-conjug

~~When~~

Next question we want to address:

When is the semi-conjugacy a conjugacy?

~~Th~~ Definition. A ~~map~~ homeomorphism $f: X \circlearrowleft$ is minimal if every orbit is dense. ~~It is topologically transitive if there exists a dense orbit. (Provisional definitions of top. trans.)~~

Theorem. $f: \mathbb{R}/\mathbb{Z} \circlearrowleft$ is minimal if and only if it is topologically conjugate to an irrational rotation.

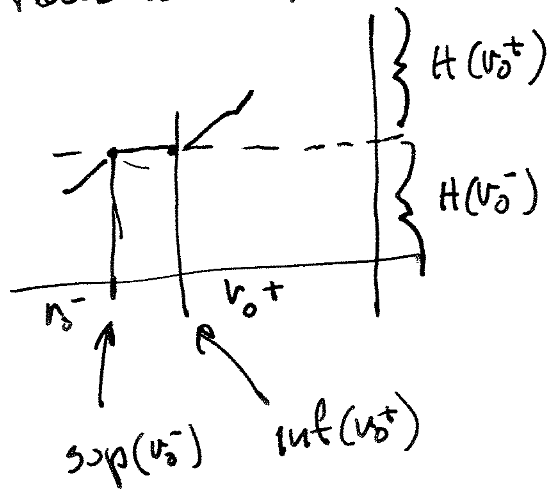
Proof. If f is minimal then $\rho(f)$ is irrational since $\rho(f)$ rational implies the existence of periodic points.

By the previous result f is semi-conjugate to a rotation. Is it possible that two distinct points map to the same point?

Discussion of extension ~~top~~ of H to $\bar{H}$ from page 5.

If $\bar{H}$ is not injective it means that there is some $v_0 \in \mathbb{R} - \Lambda_1$ so that $\sup(v_0^-)$ is strictly less than $\inf(v_0^+)$ (as defined on p. 5).

~~This means that the~~ If this is the case then every point in the interval $[\sup(v_0^-), \inf(v_0^+)]$ maps to the same point under



In particular ~~this~~ since $v_0^- \cup v_0^+ = \Lambda_1$ this means that Λ_1 is not dense.

This proves:

Proposition. If every orbit for $f: \mathbb{R}/\mathbb{Z}$ is dense then $\bar{H}$ is a ~~con~~ topological conjugacy.

(Go to page 6 definition of minimality.)

(Statement of Poincaré Thm for minimal maps)

meagre $\Leftrightarrow$ 1st category

non-meagre $\Leftrightarrow$ 2nd category

residual $\Leftrightarrow$ countable int. of dense open sets

generic contains a dense open set.