The support class for this module meets at 3pm on Mondays. This will start one week late on Oct. 19 instead of Oct. 12. Notes available on my personal website under MA424.

Dynamical systems deals with the long term behavior of simple systems.

We are interested in systems which are deterministic, finite dimensional and autonomous.

Deterministic: Current state of the system determines the state of the system at all times

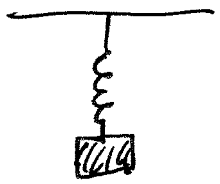
Finite dimensional: The state of the system is determined by finitely many parameters.

Autonomous: The rules governing the system do not change with time.

The systems we consider can come from a variety of fields, physics, biology, engineering or maths. If you have taken MA254 or MA371 you have seen examples.

One source of systems is ODE. Here is a simple example:

Harmonic oscillator



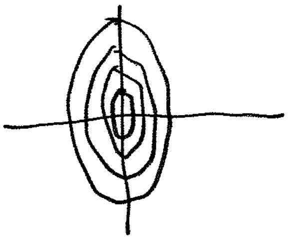
The system is modelled by the ODE $\ddot{x}(t) = -kx(t)$ where x is the deviation from rest position and k is related to the weight of the mass of the object and the strength of the spring.

The system is deterministic in that knowing the position and velocity now determines position and velocity for all time. Rewrite our equation as a first order ODE in these variables

$$\dot{x}(t) = y(t)$$
$$\dot{y}(t) = -kx(t).$$

We call $\mathbb{R}^2 = \{(x,y)\}$ phase space.

Let $\phi_t(x_0,y_0)$ be the solution with $\phi_0(x_0,y_0) = (x_0,y_0)$.



The orbit of $p = (x_0,y_0)$ is the set

$$\{\phi_t(x_0,y_0) : t \in \mathbb{R}\}.$$

In this case the orbits are ellipses that fill up $\mathbb{R}^2$.

In this case the orbits are also periodic

$\cancel{\phi_t(x_0,y_0)}$ $\phi_\omega(x_0,y_0) = (x_0,y_0)$ where $\omega = \frac{2\pi}{\sqrt{k}}$.

If we had a friction term  then orbits would be spirals. Not periodic

~~Historically such systems~~

The fact that the system is autonomous is reflected in the fact that the coefficient functions do not depend on time.

Given an autonomous system of ODE's there are two methods of approaching it. We can start with $p \in X$ and analyze $\phi_t(p)$ as t changes. This is particularly useful when we have an explicit solution. Alternatively we can fix t and think of $p \mapsto \phi_t(p)$ as a map from $X \to X$. This shift in perspective allows us to bring to bear different tools and allows us to address ~~a different~~ new questions.

"$\mathbb{R}^n$"

Write $f^t : X \to X$ for the map $f^t(p) = \phi_t(p)$.

Thm. Consider an autonomous first order ODE in n variables with C^k coefficients so that solutions are defined for all time then

⓪ $f^s : X \to X$ is C^k

① $f^{s+t} = f^s \circ f^t$

② $f^0 = Id_X$.

Cor. f^s is a C^k diffeomorphism with $(f^s)^{-1} = f^{-s}$.

Proof $\quad f^s \circ f^{-s} = f^{s-s} = f^0 = \mathrm{Id}$

$\quad\quad f^{-s} \circ f^s = f^{-s+s} = f^0 = \mathrm{Id}$.

Def. $\{f^s, X\}$ satisfying ① and ② is called a flow or an action of $\mathbb{R}$ on X.

Example: The logistic equation.

Say that $x_n \in \mathbb{R}$ represents the population of an insect species after n generations. ~~at~~ limited resources

Say that $x_{n+1} = F(x_n)$ where $F(x) = \underset{\text{growth}}{ax} - b\cancel{x}x^2$.

Then the population after n years is $F^n(x_0)$

Autonomous here means we use the same F each year. We have $x_n = F^n(x_0)$. If $X = \mathbb{R}$ (or $\mathbb{R}_+$) we get an action of the semi-group $\mathbb{Z}_+$ $= \{n \in \mathbb{Z} : n \geq 0\}$ on X

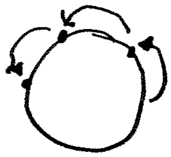
ie a map $f : \mathbb{Z}^+ \times X \to X$ satisfying (1) and (2).

Example: Rotation of the circle.

Let $X = \{z \in \mathbb{C} : |z| = 1\}$. Say $\lambda \in \mathbb{C}$ satisfies $|\lambda| = 1$.

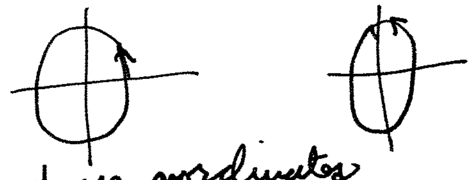
Let $M_\lambda : X \to X$ be defined by $M_\lambda(z) = \lambda z$.

$f : \mathbb{Z} \times X \to X$ given by $f^n(z) = M_\lambda^n$ defines an action of $\mathbb{Z}$ on X,



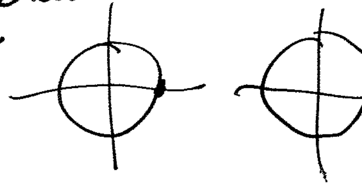
This example arises in number theory, questions of rational approximation of irrational numbers.

This last example is closely related to a system of 2 harmonic oscilators with $k_1 \neq k_2$.



Want to know how closely ~~correlated~~ the two oscilators are.

Can choose coordinates so that we have 2 circles.



Move around them at different rates. Set up a strobe light which goes off whenever the first oscilator returns to 0.

~~The effect on the second circle after~~
After ~~then~~ we rotate once around the first circle the "phase" of the second oscelator multiplies by a fixed z. After n revolutions the position of the second oscelator corresponds to M_z^n(initial position).
(We will return to this later)
In this course dynamical systems will mean semi-group actions of G on X where G is $\mathbb{Z}$, $\mathbb{Z}_+$ or $\mathbb{R}$. We will often exploit differentiability properties. ~~If~~ ~~To begin with X will be a circle~~
In many cases X is a circle S^1 or a torus $S^1 \times S^1$ or $\mathbb{R}$ or $\mathbb{C}$.

Rigid rotation of the circle.

It will be useful for us to write the circle additively.

~~Let $\mathbb{R}/\mathbb{Z}$~~ Define a map $\exp: \mathbb{R} \to \mathbb{C}$ by $\exp(r) = e^{2\pi i r}$. Two real numbers r, s map to the same point in the circle if $r - s \in \mathbb{Z}$. Then $s = r+n$ so $e^{2\pi i(r+n)} = e^{2\pi i r} \cdot e^{2\pi i n} = e^{2\pi i r} = e^s$.

Write $\mathbb{R}/\mathbb{Z}$ for $\mathbb{R}$ with the equivalence relation $r \sim s$ if $r - s \in \mathbb{Z}$. Every r is equivalent to some pt in $[0,1]$. $0 \sim 1$.
Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be defined by $R_\alpha(x) = x + \alpha \mod \mathbb{Z}$
$= x + \alpha \mod 1$
$\mathbb{R}/\mathbb{Z}$ is a metric space in a natural way.
R_α is equivalent to M_z where $z = \exp(\alpha)$.

$R_\alpha^n(x) = x + n\alpha \mod 1$

$\cancel{\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}}$

$\cancel{S^1 \to S^1}$

Definition. If $f: X \to X$ is a discrete dyn. system then $p \in X$ is a periodic point if $f^n(p) = p$ for some $n > 0$.

Proposition. For $R_\alpha: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ the following are equivalent:

(1) α is rational

(2) R_α has ~~exper~~ at least one periodic point

(3) Every point in $\mathbb{R}/\mathbb{Z}$ is periodic.

Proof. (3) $\Rightarrow$ (2) Clear.

(1) $\Rightarrow$ (3) If $\alpha = m/n$ then $R^n_\alpha(x) = R^n_{m/n}(x) = x + n \cdot \frac{m}{n} \bmod 1 = x$ so every pt. is periodic.

(2) $\Rightarrow$ (1). If $R^n_\alpha(x) = x$ then $n\alpha + x = x \bmod \mathbb{Z}$ so $n\alpha = 0 \bmod \mathbb{Z}$ so $n\alpha = m$ and $\alpha = m/n$.

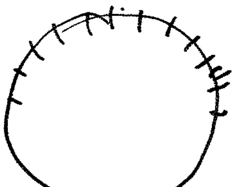
Proposition. If α is irrational then every orbit is dense in $\mathbb{R}/\mathbb{Z}$.

Proof. Assume α is irrational. Given $x \in \mathbb{R}/\mathbb{Z}$ and an open set $U \subset \mathbb{R}/\mathbb{Z}$ we need to find an n so that $R^n_\alpha(x) \in U$. Fix an $\varepsilon > 0$ so that U contains an ~~closed~~ interval I of length ε. Let $N > 1/\varepsilon$.

Consider the points $S = \{x, R_\alpha(x), R^2_\alpha(x), \dots R^N_\alpha(x)\}$.

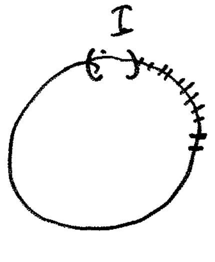
All of these points are distinct since if $R^i_\alpha(x) = R^j_\alpha(x)$ with $i < j$ then $x = R^{j-i}_\alpha(x)$ so x would be periodic and α would be rational.

Divide the circle into intervals of length $\frac{1}{N}$.



~~Use~~ By the pigeon hole principle one interval contains two distinct points in S. So $d(R^i_\alpha(x), R^j_\alpha(x)) \le \varepsilon$.

But R_α preserves distances so $d(x, R^{j-i}_\alpha(x)) < \varepsilon$.



Now the iterates of $R^{j-i}_\alpha(x)$ are less than ε apart so one of them must be in I.

Thus ~~$R^{m(j-i)}_\alpha$~~ $(R^{j-i}_\alpha)^m(x) \in I$ for some m.

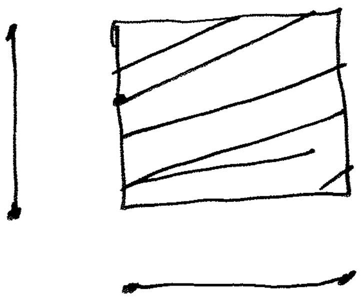
$$= R^{(j-i)m}_\alpha(x). \quad \text{Set } n = (j-i)m$$

Conclusion. The nature of the number α completely determines the dynamics.

If we consider the family of maps R_α for $\alpha \in [0,1]$ then typically α is irrational so typically every orbit of R_α is dense. since

If we think of a pair of harmonic oscillators with distinct periods this means that typically there is no relation between the observed positions.

Determinism versus "random" behavior.

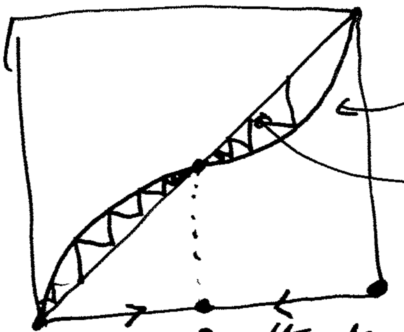


Our next topic is the dynamics of homeomorphisms of the circle. Unlike ~~the~~ rotations homeomorphisms need not preserve distance between pairs of points and this leads to some new phenomena.

If one point is periodic for a rotation then all points are periodic. For homeomorphisms this need not be the case.

Example: Let $f(x) = x + \frac{1}{10}\sin(2\pi x) \mod 1$

~~sink~~
A periodic point p is a sink if there is a nbd N of p so $q \in N$ $\Rightarrow d(f^n(p), f^n(q)) \to 0$ as $n \to \infty$.



graph of f

$x=y$

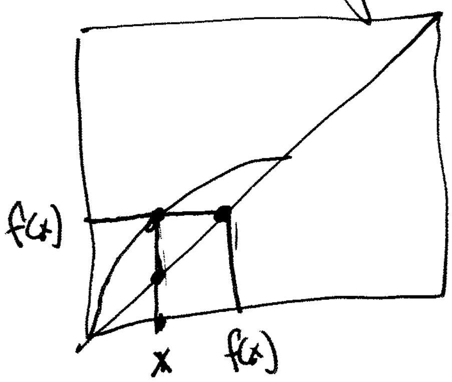
0 and $\frac{1}{2}$ are the only fixed points.

repels points in forward time: source

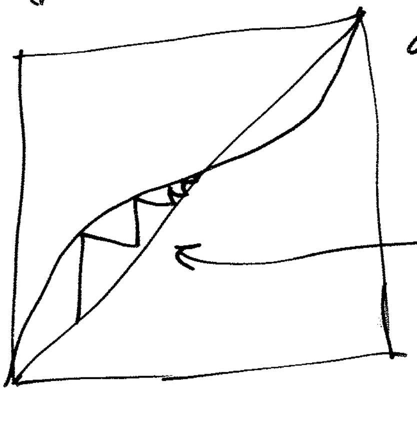
attracts pts. in forward time: sink

How do we iterate using the graph?

Graph gives a map from x variable to y variable. Diagonal gives a map from y variable back to x.



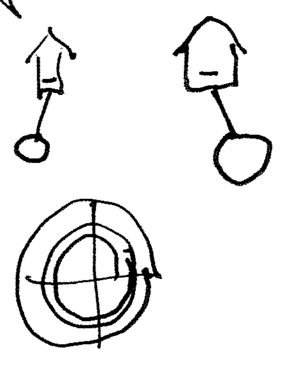
~~Go up or down to the graph then go~~
~~Go vertically~~
Start on the diagonal.
Go vertically to the graph.
Go horizontally to the diagonal.

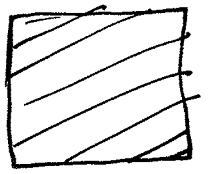


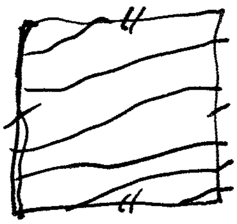
Think of the diagonal as representing the phase space!

We discussed earlier how the dynamics of R_+ related to a pair of harmonic oscillators. What about homeomorphisms? Homeomorphisms arise in the study of "weakly coupled" oscillators.

~~Say we have~~ Say we have two stable oscillators with some weak interaction. Different periods.

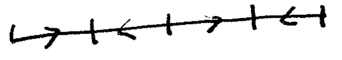




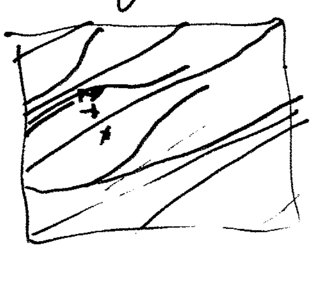


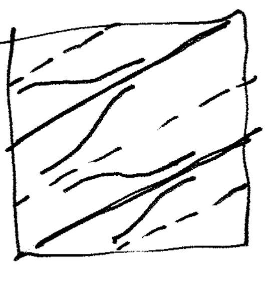
~~Ab~~ ~~A~~ Flow still corresponds to a flow on a torus. We can consider the first return map to the left hand edge. In this case it is a ~~diffeomo~~ diffeomorphism of the circle.

New phenomenon: ~~stable periodic~~ Periodic sinks.

 or . Periodic sinks give us orbits which approach periodic ~~pt~~ orbits in forward time.

"Phase locking".



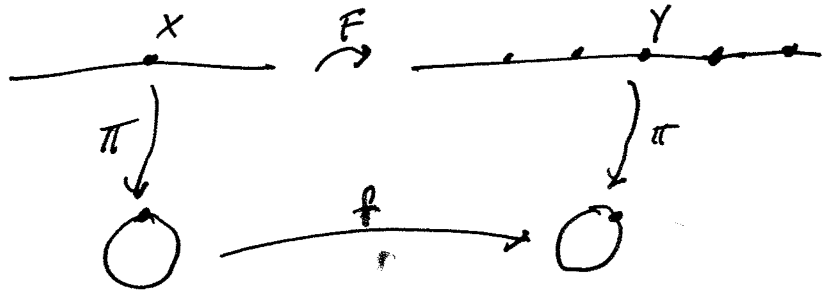


Periods of the two oscillators are rationally related.

This can be a stable condition in this setting as opposed to the case of the uncoupled oscillators.

Recall that a lift $F:\mathbb{R}\to\mathbb{R}$ of a map $f:\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ is a map such that $f(\pi(x)) = \pi(F(x))$.

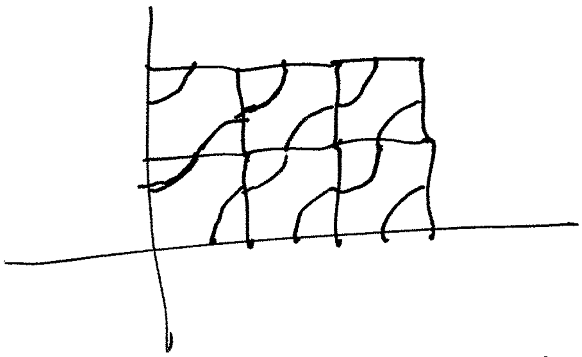
$$\begin{tikzcd} \mathbb{R} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathbb{R} \arrow[d, "\pi"] \\ \mathbb{R}/\mathbb{Z} \arrow[r, "f"] & \mathbb{R}/\mathbb{Z} \end{tikzcd}$$



Given $x\in\mathbb{R}$, ~~$F(x)=y$ satisfies the equation~~ the possible values for $y=F(x)$ differ from one another by integers.

Thm. Lifts of f exist.
Proof depends on the observation that if we have a particular x, y so that $f\pi(x)=\pi(y)$ then we can construct a lift of f in a nbd. of x. We build the lift by piecing together these local ~~f~~ lifts.

Interesting to picture, for each x the set of all possible y's so that $f\pi(x)=\pi(y)$: $\{(x,y): f\pi(x)=\pi(y)\}$



A lift ~~is a~~ of f is a subset of this picture which is a graph of a continuous function

The lift of f is built locally but captures some global information.

Proposition. ① If F and G are lifts of f then $G(x) = F(x) + k$ for some fixed $k \in \mathbb{Z}$.

② ~~[illegible]~~ $F(x+1) = F(x) + d$ for some fixed $d \in \mathbb{Z}$.

Proof. ① $F(x) - G(x)$ satisfies $\pi(F(x) - G(x)) = \pi(F(x)) - \pi(G(x))$ $= f(x) - f(x) = 0 \bmod \mathbb{Z}$ so $F(x) - G(x) \in \mathbb{Z}$ but $F(x) - G(x)$ is a continuous function so it must be constant

② $\pi\big(F(x+1) - F(x)\big) = \pi(F(x+1)) - \pi(F(x)) = f(x) - f(x) = 0 \bmod \mathbb{Z}$.
As before it must be constant

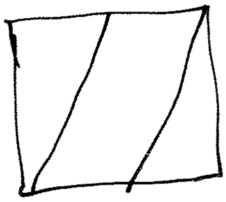
Def. The degree of $f : \mathbb{R}/\mathbb{Z} \circlearrowleft$ is $F(x+1) - F(x)$.

By ① this is independent of the lift F since $\big(G(x+1) - G(x)\big) - \big(F(x+1) - F(x)\big) = \big(G(x+1) - F(x+1)\big)$
$- \big(G(x) - F(x)\big) = k - k = 0$.

By ② this number is independent of x.

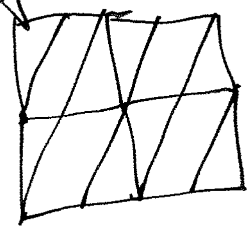
Example ①. $f(x) = 2x \mod 1$.

Note that this is well defined since $f(x+k) = 2(x+k) = 2x + 2k$
$= 2x \mod 1$.

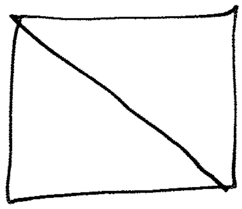


$F(x) = 2x$ is a lift of f.

$\deg f = F(1) - F(0) = 2\cdot 1 - 2 \cdot 0 = 2$.



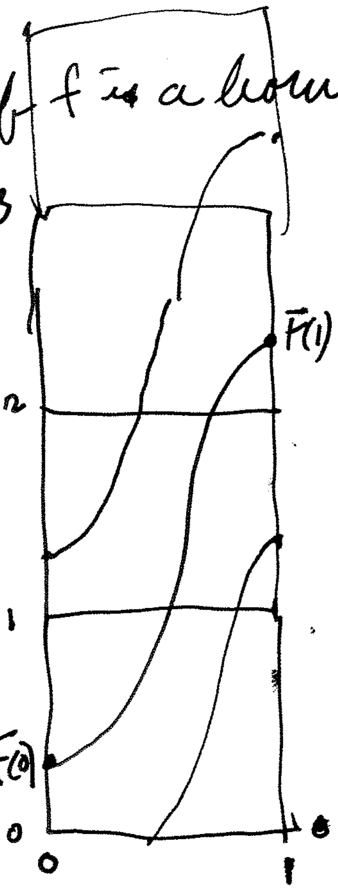
② $f(x) = -x \mod 1$



Example 2 is a homeomorphism while example 1 is not.

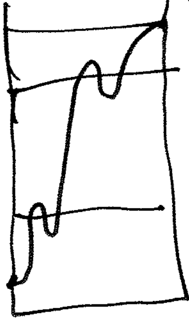
Prop. If f is a homeomorphism then $\deg f = \pm 1$.

Proof.



Degree 2. $f^{-1}(0)$ has 2 (or more) points so f is not injective.

~~Same argument shows~~



$\overset{x}{\shortparallel}\ F^{-1}(1)$ and $\overset{x'}{\shortparallel}\ F^{-1}(2)$ both satisfy

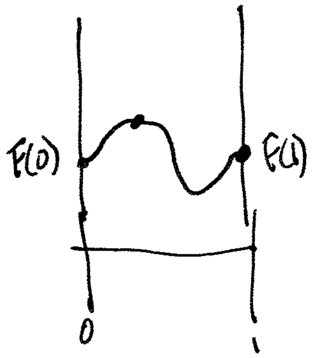
~~$f(x) = f(x') \mod \mathbb{Z}$~~

~~$f(x) = f(x') \Leftrightarrow F(x) = F(x')$~~

$f(x) = F(x) \mod 1 = 1 \mod 1 = 0$

$f(x') = F(x') \mod 1 = 2 \mod 1$

Same argument shows $|\deg f| \le 1$ so $\deg f = -1, 0, 1$.



$\deg f = 0$ ~~violates the mean va~~ cannot be injective due to the mean value thm.

~~The case of degree 1 corresponds to orien~~

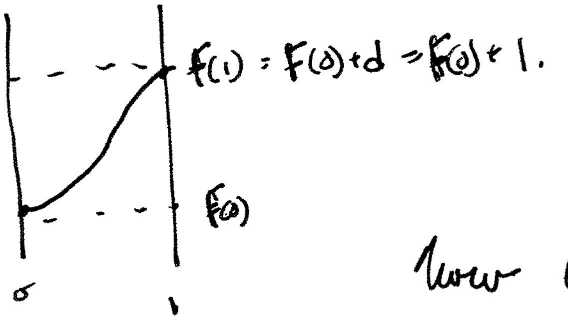
What is different about $\deg f = 1$ and $\deg f = -1$?

~~There is~~ In terms of the lift F $\deg f = 1$ corresponds to a monotone increasing function, $\deg f = -1$ corresponds to monotone decreasing

~~This can be stated in terms of orientation of the circle.~~

Prop. If f is a homeomorphism with degree 1 then F is monotone increasing.

Proof.



$F([0,1]) = [F(0), F(1)]$

Now $F([n, n+1])$

Now

$= [F(n), F(n+1)] = [F(0)+n, F(1)+n]$

using $F(x+1) = F(x)+1$ repeatedly.

Lemma. Let F be a monotone increasing function from $\mathbb{R}$ to $\mathbb{R}$ (for example a lift of a degree 1 circle homeomorphism). If $F(x) \geq x$ then $F^k(x) \geq x$ for all $k > 0$. If $F(x) \leq x$ then $F^k(x) \leq x$ for $k \geq 0$.

Assume $F(x) \geq x$

Proof. We prove that $x \leq F(x) \leq F^2(x) \leq \ldots \leq F^k(x)$ (*) by induction on k. Statement is true for $k=1$ by assumption. Assume the statement for k so $x \leq F(x) \leq \ldots F^k(x)$. By monotonicity we have $F(x) \leq F^2(x) \leq \ldots F^k(x) \leq F^{k+1}(x)$. Combine this with $x \leq F(x)$ and we have our statement for $k+1$.

Our next objective is to show that $\lim_{n\to\infty} \frac{F^n(x)}{n}$ exists. We will use repeatedly the fact that $F(x+1) = F(x)+1$. We can rewrite this. Let $T(x) = x+1$. Then we have ~~ELL~~ $F \circ T = T \circ F$ since $F \circ T(x) = F(x+1)$ and $T \circ F(x) = F(x)+1$.

(This can be cast in terms of a $\mathbb{Z}^2$ action on $\mathbb{R}$).

Example: $(T^n \circ F^m)^p = (\underbrace{T^n \circ F^m \circ T^n \circ F^m \circ \quad T^n \circ F^m}_{p})$

$= T^{pn} \circ F^{pm}$

We would like to capture the behavior of f with a number that plays a role like the α in R_α. α is the increase in the x coordinate of a lift of R_α. In general the increase in the x coordinate of a lift F of f is not constant so we consider the average increase along the orbit.

Let $f : \mathbb{R}/\mathbb{Z} \circlearrowleft$ and F be a lift

Def. $\rho(F) = \lim\limits_{n \to \infty} \dfrac{F^n(x)}{n}$

and $\rho(f) = \rho(F) \bmod 1$.

We call $\rho(f)$ the rotation number of f.

In order to make this well defined we need to show that $\rho(F)$ is independent of x and $\rho(f)$ is independent of the choice of the lift.

Example. $\cancel{\rho(R_\alpha) = \alpha}$ $f = R_\alpha$, $F = T_{\alpha+k}$ where

$T_{\alpha+k}(x) = x + \alpha + k$. Then $\rho(T_{\alpha+k}) = \lim\limits_{n \to \infty} \dfrac{n(\alpha+k)+x}{n} = \alpha + k$.

$\rho(R_\alpha) = \alpha + k \bmod 1$.

Lemma. $\lim_{n \to \infty} \frac{F^n(0)}{n}$ exists.

Proof. Say that $F^n(0) \in [k, k+1]$ for some $k \in \mathbb{Z}$. Then $F^n(0) \geq k$ so $T^{-k} F^n(0) \geq 0$.

$$(\underbrace{T^{-k} F^n}_{\text{monotone increasing}})^m (0) \geq 0 \text{ for all } m.$$

Do we really need to show that $T^{-k}F^n$ is a lift? Suffices to know $T^{-k}F^n$ is monotone increasing.

Thus $T^{-km} F^{nm}(0) \geq 0$

so $F^{nm}(0) \geq T^{km}(0) = km$ for all m

· ~~[illegible] $F^{nm}(0)$~~

Similarly if $F^n(0) \leq k+1$ so $F^{nm}(0) \leq (k+1)m$

$$T^{-k-1} F^n(0) \leq 0 \to (T^{-k-1} F^n)^m (0) \leq 0 \to T^{-m(k+1)} F^{nm}(0) \leq 0$$

Thus $km \leq F^{nm}(0) \leq (k+1)m$

so

$$\frac{km}{mn} \leq \frac{F^{nm}(0)}{mn} \leq \frac{(k+1)m}{mn}$$

$$\frac{k}{n} \leq \frac{F^{nm}(0)}{mn} \leq \frac{(k+1)}{n}$$

$$\frac{F^{nm}(0)}{mn} \in \left[\frac{k}{n}, \frac{k+1}{n}\right].$$

Pretty good estimate but only for a subsequence.

True for all $\underline{m}$.

$$\left| \frac{F^n(0)}{n} - \frac{F^{nm}(0)}{nm} \right| \leq \frac{1}{n}$$

So for any $n, m \in \mathbb{N}$ we get

$$\left| \frac{F^n(0)}{n} - \frac{F^m(0)}{m} \right| \leq \left| \frac{F^n(0)}{n} - \frac{F^{nm}(0)}{nm} \right| + \left| \frac{F^m(0)}{m} - \frac{F^{nm}(0)}{nm} \right| \leq \frac{1}{n} + \frac{1}{m}.$$

add and substract $\frac{F^{nm}(0)}{nm}$

This shows that $\frac{F^n(0)}{n}$ is a Cauchy sequence:

Take $\varepsilon > 0$. For $n, m > \frac{2}{\varepsilon}$ we get

$$\left| \frac{F^n(0)}{n} - \frac{F^m(0)}{m} \right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

In particular $\frac{F^n(0)}{n}$ converges.

~~Remark: Could have started with~~ ~~x~~ ~~instead of~~ ~~a_k~~

~~Get if~~ ~~$F^n(x) \in [x+k, x+k+1]$~~ ~~then~~ ~~$\frac{F^{nm}(x)}{nm} \in$~~

Scholium: If $F^n(0) \geq k$ then $\rho(F) \geq \frac{k}{n}$.

~~If~~ ~~$F^n(x) - x \geq k$~~ ~~then~~ ~~$\rho(F) \geq \frac{k}{n}$~~

~~If~~ ~~$F^n(0) \leq k$~~ ~~then~~ ~~$\rho(F) \leq \frac{k}{n}$~~ Uses convergence.

Uses $\rho(F) = \lim_{m \to \infty} \frac{F^{nm}(0)}{nm} \geq \frac{k}{n}$.

Also $F^n(x) - x \geq k$ then $\rho(F) \geq \frac{k}{n}$.

Lemma. $\lim_{n\to\infty} \frac{F^n(x)}{n}$ exists for all x and takes the same value.

Proof. For some $m \in \mathbb{N}$ $\quad -m \le x \le m$.

Thus $F^n(-m) \le F^n(x) \le F^n(m)$ since F^n is monotone increasing

$$F^n T^{-m}(0) \le F^n(x) \le F^n T^m(0)$$

$$T^{-m} F^n(0) \le F^n(x) \le T^m F^n(0)$$

$$F^n(0) - m \le F^n(x) \le F^n(0) + m$$

$$\frac{F^n(0)-m}{n} \le \frac{F^n(x)}{n} \le \frac{F^n(0)+m}{n}.$$

So $\lim_{n\to\infty} \frac{F^n(x)}{n} = \lim_{n\to\infty} \frac{F^n(0)}{n}$.

Lemma. Let F and G be lifts of f Then

$\rho(F) = \rho(G) \mod 1$.

Proof. $F(x) = G(x) + k$ for some k.

$$\lim_{n\to\infty} \frac{F^n(x)}{n} = \lim_{n\to\infty} \frac{(G(x)+k)^n}{n} = \lim_{n\to\infty} \frac{(T^k G)^n(x)}{n}$$

$$= \lim_{n\to\infty} \frac{T^{kn} G^n(x)}{n}$$

$$= \lim_{n\to\infty} \frac{G^n(x) + kn}{n}$$

$$= \lim_{n\to\infty} \frac{G^n(x)}{n} + k.$$

how the rotation number is defined for homeomorphisms deg 1

Our objective is to connect properties of the rotation number $\rho(f)$ to dynamical properties of f.

Let's look at rotation numbers.

Proposition. Let f be an orientation preserving homeomorphism of the circle then $\rho(f)=0$ if and only if f has a fixed point.

Proof. Assume $p=f(p)$. Let F be a lift of f.
~~Since~~ $F(p)$ and $\mathbb{Z} = f(p) = p$ so $F(p)-p \in \mathbb{Z}$. Say $F(p)-p=k$. Then ~~F(x)~~ $G(x)=F(x)-k$ is also a lift of f and $G(p)=F(p)-k=p$.
Now $\rho(f) \equiv \rho(G) = \lim_{n\to\infty} \frac{G^n(p)}{n} = \lim_{n\to\infty} \frac{p}{n} = 0$. So $\rho(f)=0$.

~~Now suppose~~
Now we prove the converse. Assume that $\rho(f)=0$. Let F be a lift. We have $\rho(F)=k$. Let $G(x)=F(x)-k$ so $\rho(G)=0$.
Now $G(0)=0$ or $G(0)<0$ or $G(0)>0$. If $G(0)=0$ we are done. Assume $G(0)>0$. Then ~~G(0)~~ $G^n(0)$ is an increasing sequence. Say $G^n(0)>1$ for some n. Then

according to our scholium we have $\rho(G) \geq \frac{1}{n}$.
~~Bu~~ This would contradict our assumption.
So $G^n(0)$ is increasing and bounded. Let

$\lim_{n\to\infty} G^n(0) = x^*$. Then ~~$\lim_{n\to\infty} G^n(G^n(0)) = G(x^*)$~~

$$G(x^*) = \lim_{n\to\infty} G(G^n(0)) = \lim_{n\to\infty} G^{n+1}(0) = x^*.$$

So x^* is fixed by G and $\pi(x^*)$ is fixed by f.

Prop. $\rho(f^n) = n\,\rho(f)$.

Proof. Let F be a lift of f. Then F^n is a lift of f^n.

$\cancel{\rho(f^n) = \lim}$

$$\rho(F^n) = \lim_{m\to\infty} \frac{F^{nm}(x)}{m} = n \lim_{n\to\infty} \frac{F^{nm}(x)}{nm} = n\cdot\rho(F).$$

Reducing mod 1 gives:

$$\rho(f^n) = n\,\rho(f).$$

$\boxed{\text{True for } n<0\text{. Follows from } \rho(f^{-1}) = -\rho(f)\text{. Exercise.}}$

Theorem. f has a periodic ~~point~~ orbit if and only f has a rational rotation number.

Proof. If f has a periodic orbit of period n then $f^n(p) = p$ so $\rho(f^n) \equiv 0 \bmod 1$. If $\rho(f) = \frac{m}{n}$ then $\rho(f^n) = n\cdot\frac{m}{n} = m = 0 \bmod 1$ so f^n has a fixed point.

For future use we want to extract a little more from our lemma on rotation numbers.

Lemma. If f is an orientation preserving homeomorphism of $\mathbb{R}/\mathbb{Z}$ and F is its lift then if $F^n(x) \geq x+k$ we have $n\rho(F) \geq k$ for $n \in \mathbb{Z}$.

Proof. Assume $F^n(x) \geq x+k$

so $$F^n(x) \geq T^k(x)$$

$$T^{-k}F^n(x) \geq x$$

Using previous lemma:

$$\left(T^{-k}F^n\right)^m(x) \geq x \qquad \text{for any } m>0$$

$$F^{nm}(x) \geq x + km$$

$$\frac{F^{nm}(x)}{m} \geq \frac{x+km}{m}$$

Let $m \to \infty$ $$\rho(F^n) \geq k$$

so $$n\rho(F) \geq k.$$

A natural notion of equivalence in dynamical systems.

We say that $f: X \circlearrowleft$ and $g: Y \circlearrowleft$ are topologically conjugate if there is a homeomorphism $h: X \to Y$ so that $g \circ h = h \circ f$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

Why is this a good notion?

Behaviours we are interested in are topological:
x has a dense orbit, x is periodic, x is attracted to a periodic orbit.

If a point $x \in X$ has a given ~~behav~~ topological behaviour
$p \in X$
in X (it is dense say) then $q = h(p)$ has the same topological behaviour in Y.

h takes orbits to orbits: ~~hff~~ $h(f^n(p)) =$
$hff\ldots f(p) = ghf\ldots f(p) = g\ldots g\, h(p) = g^n(q)$.

Since h is a homeomorphism it takes dense sets to dense sets.

Fits in with the ~~philosophy~~ idea of answering not ~~how~~ does $g^n(\xi)$ ~~look~~ $f^n(p)$ behave for a given p but rather given some behavior ~~for~~ is there ~~a~~ some point p that realizes it?

Can also define a semi-conjugacy where we do not assume h is invertible. Gives partial information. Not an equiv. relation.

~~Proposition~~

~~The rotation number~~

~~A quantity attached~~ A quantity attached to a ~~...~~ dynamical system is a dynamical invariant if it takes the same value on top. conjugate systems. The rotation number is almost a topological invariant.

Proposition. Let circle homeomorphisms be topologically conjugate via a homeo. h. Then if $\deg h = 1$ $\rho_f = \rho_g$. If $\deg h = -1$, $\rho_f = -\rho_g$.

Proof. If $g \circ h = h \circ f$ then $g = h \circ f \circ h^{-1}$. Let H be a ~~f~~ lift of H and F a lift of f then $G = H \circ F \circ H^{-1}$ is a lift of g. Let $x = H(0)$

$$\cancel{\frac{G^n(x)}{n}} \quad \frac{G^n(x)}{n} = \frac{H \circ F^n \circ H^{-1}(x)}{n} = \frac{H \circ F^n(0)}{n}$$

Say $F^n(0) \in [k, k+1]$ then $T^{-k} F^n(0) \in [0, 1]$

$\boxed{H(x+1) = H(x)+d}$ $HT = T^d H$ where (16)
~~k~~ depends on deg H.
$d = \pm 1$.

$$\frac{H \circ F^n(0)}{n} = \cancel{H(T^{-k} F^n(0)} \quad \frac{H(T^k T^{-k} F^n(0))}{n} = \frac{T^{\pm k} H T^{k} F^n(0)}{n}$$

Now if $T^{-k}F^n(0) \in [0,1]$ then $HT^kF^n(0) \in H([0,1])$

$$= \frac{HT^{-k}F^n(0) \pm k_n}{n} \in$$

$$\rho(G) = \lim_{n\to\infty} \frac{G^n(x)}{n} = \pm \lim_{n\to\infty} \frac{k_n}{n} = \lim_{n\to\infty} \frac{F^n(0)}{n} = \pm \rho(F).$$

Cor. Two rotations R_α, R_β are top. conjugate iff $\alpha = \pm\beta$.

Proof. Any map is top. conj. to itself. If $\alpha = -\beta$ then let $h(x) = -x$ mod 1 so ~~R_α~~ $h^{-1} R_\beta h(x) = -(\beta + -x) = x - \beta = R_\alpha(x)$.

On the other hand if R_α and R_β are top. conjugate then $\rho(R_\alpha) = \pm\rho(R_\beta)$ so $\alpha = \pm\beta$.

To what extent does rotation number capture ~~the~~ the idea of top. conjugacy?

~~If f has rational rotation num~~
If f has a periodic sink then f is not top equiv. to any R_α. Of course this can only happen if $\rho(f)$ is rational.

Last time we talked about the notion of topological conjugacy between dynamical systems.

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow h & & \downarrow h \\ Y & \xrightarrow[g]{} & Y \end{array}$$

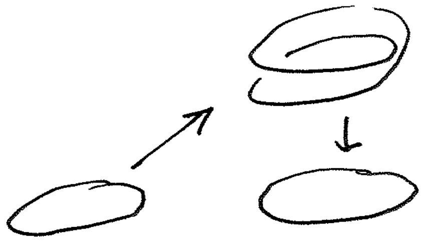
Sometimes useful to talk about a weaker notion of semi-conjugacy.

We say there is a semi-conjugacy ~~between~~ from (f,X) to (g,Y) if there is a continuous h that makes the diagram commute.

~~Here~~ In this case (g,Y) reflects some of the dynamics of (f,X) but perhaps not all since h can identify distinct points.

Example. There is a semi-conjugacy between $R_\alpha : \mathbb{R}/\mathbb{Z} \circlearrowleft$ and $R_{2\alpha} : \mathbb{R}/\mathbb{Z} \circlearrowleft$ given by $h(x) = 2x \mod 1$.

$$h \circ R_\alpha(x) = R_{2\alpha} \circ h(x)$$
$$\begin{array}{ccc} \| & & \| \\ 2(x+\alpha) & & 2x + 2\alpha. \end{array}$$



In this case h is 2-1. Note that these maps are not ~~conju~~ top. conjugate unless $\alpha = \pm 2\alpha \mod 1$. Since the $\pm$ rotation number is a topological invariant.

Question: If f, g have the same irrational rotation number are they top. conjugate?

Theorem. If $\rho(f) = \alpha$ is irrational then ~~f is semi-conjugate to R_α.~~ there is a semi-conjugacy from f to R_α.

~~Think of $x \in$~~ $\mathbb{R} \to S^1$

Proof. Let $G(x) = x + \alpha$. G is a lift of R_α. ~~So $\rho(G) = \alpha$.~~

~~Let F be a lift of f with $\rho(F) = \alpha$.~~

~~(Recall that we can adjust $\rho(F)$ by an integer)~~

Let F be a lift of f with $\rho(F) = \alpha_0$ $(\in \mathbb{R})$ $\alpha_0 \equiv \alpha$ mod 1

Let $G(x) = x + \alpha_0$. So $\rho(F) = \rho(G) = \alpha_0$.

Pick $x_0 \in \mathbb{R}$

Let $\Lambda_1 = \{ F^n(x_0) + m \,;\, m, n \in \mathbb{Z} \}$

$\Lambda_2 = \{ G^n(0) + m \,;\, m, n \in \mathbb{Z} \}$. $\qquad T^m F^n(x_0)$ $\shortparallel$

Consider the maps ~~$L_1(m,n)$~~ $L_1 : \mathbb{Z}^2 \to \Lambda_1$, $L_1(m,n) = F^n(x_0) + m$.

$L_2 : \mathbb{Z}^2 \to \Lambda_2$, $L_2(m,n) = G^n(0) + m$. $\qquad \shortparallel$

Claim: L_1, L_2 are injective. ~~(Typically with dense images.)~~ $T^m G^n(0)$ (The image of L_2 is dense. Follows from denseness of orbits of R_α).

If $L_1(m,n) = L_1(m',n')$ then $F^n(x_0) + m = F^{n'}(x_0) + m'$

so, reducing mod 1, $f^n(x_0) = f^{n'}(x_0)$ mod 1.

This gives $f^n(x_0) = f^{n'}(x_0)$ reducing mod 1 or

$f^{n-n'}(x_0) = x_0$. Since f has no periodic points

we have $n - n' = 0$ or $n = n'$. ~~$T^m F^n(x_0) = T^{m'} F^n(x_0)$~~

$F^n(x_0) + m = F^n(x_0) + m' \Rightarrow m = m'$.

Lemma (with altered notation).

Let f be a homeomorphism with $\rho(f) = \rho$ irrational. Let F be a lift of f with $\rho(F) = \rho_0$.

Let $G(x) = x + \rho_0$ so G is a lift of R_ρ. Pick $x_0 \in \mathbb{R}$.

Let $\Lambda_1 = \{ F^n(x_0) + m ;\ m, n \in \mathbb{Z} \}$

$$\Lambda_2 = \{ \underbrace{G^n(0)}_{= n\rho_0} + m ;\ m, n \in \mathbb{Z} \}$$

$n\rho_0 + m$

Define $H : \Lambda_1 \to \Lambda_2$ by $H(F^n(x_0) + m) = n\rho_0 + m$.

Then H is bijective, strictly increasing and $H(x+1) = H(x) + 1$

$HF(x) = GH(x)$. $(= H(x) + \rho_0)$.

Proof of Lemma. Consider the maps

$L_1(m,n) = F^n(x_0)+m$ $\quad (= T^m F^n(x_0))$ $\quad L_1 : \mathbb{Z}^2 \to \Lambda_1$

$L_2(m,n) = G^n(0)+m$ $\quad (= T^m G^n(0))$ $\quad L_2 : \mathbb{Z}^2 \to \Lambda_2$.

If $L_1(m,n) = L_1(m',n')$ then $F^n(x_0)+m = F^{n'}(x_0)+m'$ (*)

so, reducing mod 1, $f^n(x_0) = f^{n'}(x_0) \mod 1$.

This gives $f^{n-n'}(x_0)=x_0$ and $n-n'=0$ since f

has no periodic points. Follows from (*) that $m=n'$!

Same argument shows L_2 is injective so

$H = L_2 \circ L_1^{-1}$ is injective.

Now take $x_1, x_2 \in \Lambda_1$ with $x_1 < x_2$.

$x_1 = F^{n_1}(x_0)+m_1 < F^{n_2}(x_0)+m_2 = x_2$ for some n_1, n_2, m_1, m_2.

~~Let $y = F^{n_2}(x_0)$. Then $F^{-n_2}(y) = x_0$~~

$F^{n_1-n_2}(x_0) < x_0 + m_2 - m_1$ (recall F, T commute)

$\rho^{n_1-n_2} \le m_2 - m_1$ equality not possible.

~~If γ were 0 then we could something about ρ and hence σ.~~

Thm. If f is minimal it is conjugate to a rotation.

Poincaré's

Proof of theorem. Since f is minimal f has no periodic points. Consequently ~~ρ~~ $\rho(f)$ is irrational. Let F be a lift of f, $x_0 \in \mathbb{R}$ and

$\Lambda_1 = \Lambda_{x_0} = \{F^n(x_0) + m\}$

Let $\rho_0 = \rho(f)$. $\rho_0 \in \mathbb{R}$.

Let $\Lambda_2 = \{n\rho_0 + m\} = \{G^n(0) + m\}$ where $G(x) = x + \rho_0$.

By the Lemma we have an strictly ~~monotone~~ increasing ~~orientation~~ preserving map from $\Lambda_1 \subset \mathbb{R} \xrightarrow{H} \Lambda_2 \subset \mathbb{R}$.

We can extend H to a $\bar{H}: \mathbb{R} \to \mathbb{R}$ as follows.

If $v_0 \in \mathbb{R} - \Lambda_1$ let $v_0^- = \{\cancel{v \in \Lambda_1}\ v < v_0,\ v \in \Lambda_1\}$

$v_0^+ = \{v > v_0,\ v \in \Lambda_1\}$

Then every element of $H(v_0^-)$ is less than every element of $H(v_0^+)$ so $\sup(H(v_0^-)) \le \inf(H(v_0^+))$.

~~On the other hand~~ We can ~~clearly~~ extend H by sending v_0 to any element of $[\sup(H(v_0^-)), \inf(H(v_0^+))]$.

On the other hand the union of $H(v_0^-)$ and $H(v_0^+)$ is $H(\Lambda_1) = \Lambda_2$ so it is dense. This means $\sup(H(v_0^-)) = \inf(H(v_0^+))$.

This extension is strictly increasing and hence injective. This follows from the denseness of Λ_1. If $\bar{H}(x)=\bar{H}(y)$ then choose $\lambda_1, \lambda_2 \in \Lambda_1$ with $x < \lambda_1 < \lambda_2 < y$. We have $\bar{H}(x) \le \lambda_1 < \lambda_2 \le \bar{H}(y)$.

An increasing function is continuous. (Check that the inverse image of an interval contains a non-trivial interval.)

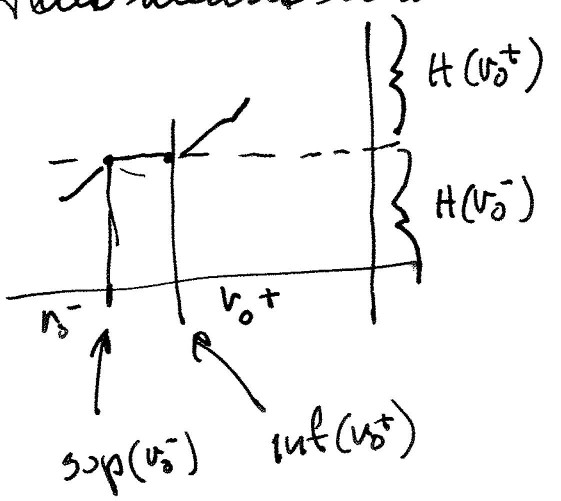
Since $H(x+1) = H(x)+1$ and $H \circ F = G \circ H$ we have

(1) $\bar{H}(x+1) = \bar{H}(x)+1$ and $\bar{H} \circ F = G \circ \bar{H}$. An increasing function satisfying (1) is a lift of a homeomorphism. Let h be that homeomorphism. Reducing mod 1 we get $h \circ f = R_\rho \circ h$ so h is the conjugacy we are seeking.

Discussion of extension ~~top~~ of H to $\bar{H}$ from page 5.

If $\bar{H}$ is not injective it means that there is some $v_0 \in \mathbb{R} - \Lambda_1$ so that $\sup(v_0^-)$ is strictly less than $\inf(v_0^+)$ (as defined on p. 5).

~~This means that the~~ If this is the case then every point in the interval $[\sup(v_0^-), \inf(v_0^+)]$ maps to the same point under H



In particular then since $v_0^- \cup v_0^+ = \Lambda_1$ this means that Λ_1 is not dense.

This proves:

Proposition. If every orbit for $f: \mathbb{R}/\mathbb{Z}$ is dense then $\bar{H}$ is a ~~con~~ topological conjugacy.

(Go to page 6 definition of minimality.)

(Statement of Poincaré Thm for minimal maps)

Meagre $\Leftrightarrow$ 1st category

non-meagre $\Leftrightarrow$ 2nd category

residual $\Leftrightarrow$ countable int. of dense open sets

generic contains a dense open set.

Two basic questions about circle homeomorphisms with irrational rotation numbers

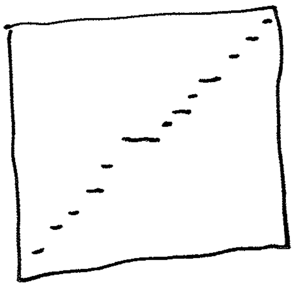
Q. ① Are all orbits dense?
② ~~Can they have orbits which are not dense~~?
② Are they topologically ~~equivalent to~~ conjugate to rotations?

According to Poincaré's theorem on conjugacies, these two questions are equivalent.

Our next project is to construct "Denjoy's example" which shows that the answer to ① and ② is "not necessarily". [To some extent the structure of this counter-example is determined by what we already know.

Poincaré's theorem on semi-conjugacies tells us that such a counter example is semi-conjugate to an irrational rotation. Thus there is an h from $\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ which is increasing but not strictly increasing and which conjugates f to R_θ.
We assume $h(0)=0$. Can think of $h: ~~I~~\ [0,1] \to [0,1]$.

graph of h.



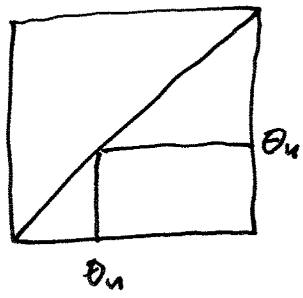
$$\begin{tikzcd} \mathbb{R}/\mathbb{Z} \arrow[r, "f"] \arrow[d, "h"] & \mathbb{R}/\mathbb{Z} \arrow[d, "h"] \\ \mathbb{R}/\mathbb{Z} \arrow[r, "R_\theta"] & \mathbb{R}/\mathbb{Z} \end{tikzcd}$$

What can we say about the image of the horizontal plateaus? If $h(x)=h(x')$ then $h(f(x))=h(f(x'))$
so if $h(I)=y$ then $h(f(I)) = R_\theta(y)$. and $h(f^n(I)) = R_\theta^n(y)$.
Let $I_n = f^n(I)$ then the image of $\bigcup_n I_n = \mathcal{O}(y)$ is dense in $\mathbb{R}/\mathbb{Z}$.

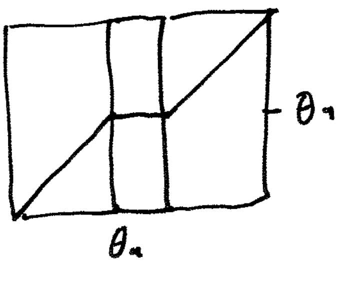
It follows that h is constant on a dense subset of the circle. Such a function is sometimes called a "Devil's staircase".

~~It~~ In order to construct f we start by constructing h. Let's assume that the image of the intervals of constancy I_n is the orbit of 0 so $h(I_n) = R_\theta^n(0) = n\theta \mod 1$. Let us call this point θ_n and think of $\theta_n \in [0,1)$. Let ℓ_n be the length of I_n. Assume $\sum \ell_n = L < \infty$.

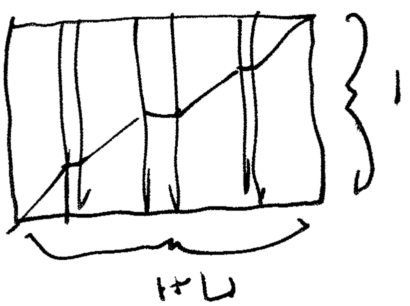
Here is a geometric construction of ~~h~~ the graph of h. Start with the diagonal in the square. For each θ_n cut the square vertically and insert a rectangle of height 1 and width ℓ_n. Make the graph ~~h~~ constant on this rectangle taking the value θ_n.



The challenging part is that we need to do this operation countably many times.



In the end we wind up with a function from $[0, 1+L]$ to $[0,1]$.



For future use we want an ~~analytic~~ formula for h. In fact it is easier to get a formula for h^{-1}.

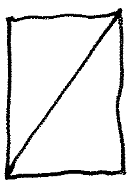
For $y \notin O(0)$ we get $\displaystyle h^{-1}(y) = y + \sum_{\theta_j \in [0,y)} l_j$.

If $y \in O(0)$ then h^{-1} has a jump type discontinuity at y where the left and right hand limits are $\displaystyle y + \sum_{\theta_j \in [0,y)} l_j$ and $\displaystyle y + \sum_{\theta_j \in [0,y]} l_j$.

It is not hard to show that h^{-1} is continuous at the remaining points.

Knowing h determines f on the complement of $\cup I_n$. If $x \notin \cup I_n$ then $R_\theta h(x) \notin O(0)$ so $h^{-1} R_\theta h(x)$ is a unique point which is the only possible value for $f(x)$.

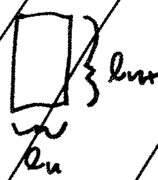
It remains to define f on $\cup I_n$. Arguing as above we see that $f(I_n)$ must be equal to I_{n+1}. If we take f to be linear on each I_n then we get a continuous homeomorphism semi-conjugate to R_θ.



I_{n+1}

I_n

We have countably many rectangles on which f is not defined.



If $x \notin \bigcup I_n$ then $f(x)$ is uniquely determined by $f(x) = h^{-1}(R_\theta(h(x)))$.
If $x \in I_n$ then $f(x) \in h^{-1}(R_\theta(h(x))) = I_{n+1}$.

We can fill in the graph in these rectangles with straight lines to produce a homeomorphism f.

Question. How ~~can~~ differentiable can we make f?

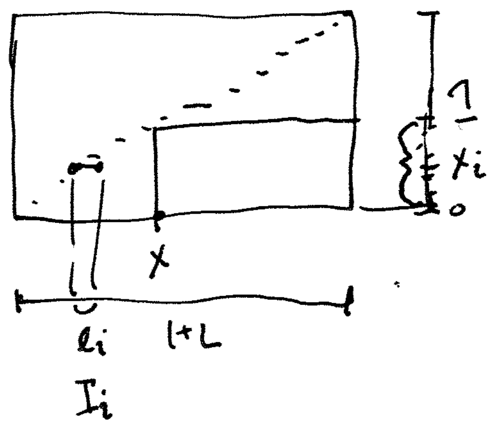
Recall that if f arises from the time one map of an ODE then the differentiability of f is the same as the differentiability of the coefficients of the ODE. ~~Something~~ we can get our hands on.
Does this differentiability influence the dynamics that can occur?

Theorem. (Denjoy) There is a C^1 diffeomorphism of the circle which ~~is~~ has irrational rotation number but which is not minimal.

Proof. The things we can control are the lengths of gaps l_n and graph of f when it maps I_n to I_{n+1}.

Assume $\frac{l_n}{l_{n+1}} \to 1$ as $n \to \pm\infty$. For example take $l_n = \frac{1}{n^2+25}$ $\quad \frac{1}{2} < \frac{l_{n+1}}{l_n} < 2$.

Say ~~$l_n = [a_n, b_n]$~~ $I_n = [a_n, b_n]$ where $b_n - a_n = l_n$.



Recall that there is a semi-conjugacy from f to the rotation R_θ.

The ~~[illegible]~~ interval of length 1+L is obtained from the interval of length 1 by replacing the point $x_i = R_\theta^i(0)$ by an interval of length l_i.

~~x maps to~~ If $x \notin O(0)$ then x maps to $h(x)$ where

$$x = h(x) + \sum_{x_i \in [0, h(x)]} l_i .$$

This is really a formula for h^{-1} at least when $x \notin O(0)$.

Write $y = h(x)$. ~~Then~~ and $h^{-1}(y) = x$

$$h^{-1}(y) = y + \sum_{x_i \in [0, y]} l_i .$$

h^{-1} is a function with jump type discontinuities at the points x_i.

skip

Want a formula for H the lift of h to $\mathbb{R}$.
Write $x_{i,j}$ for $T^j R_\theta^i(0)$. We called the set ~~[illegible]~~ of $x_{i,j}$'s Λ_2 in the context of the Poincaré thm.
~~Cl~~ Define $H(x)$ implicitly by

$$x = H(x) + \sum_{x_{i,j} \in [0, H(x)]} l_i .$$

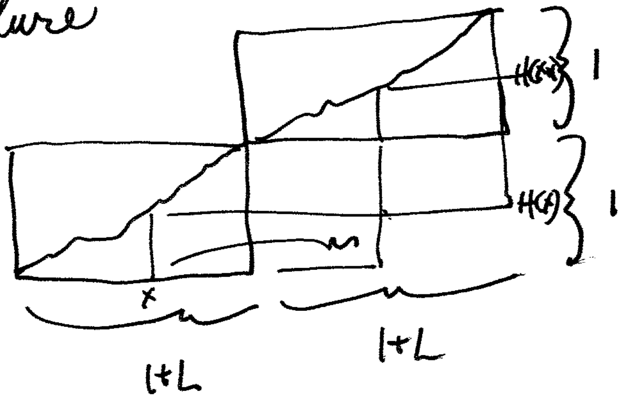
skip

Claim: ① $H(x) = h(x)$ for $x \in [0,1)$

② $H(x+L+1) = H(x)+1$

③ H is monotone.

~~① follows from~~
Picture

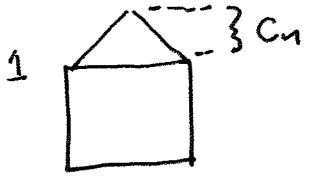


② follows from $\sum_{x_i \in [H(x), H(x+1))} l_i = L$.

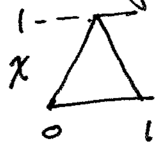
① follows from every $x_i \bmod 1$ has a unique representative as $x_{i,j}$.

We want $f|I_n$ to have a continuous first derivative and we will want f to have derivative 1 at either end of I_n. Say $I_n = [a_n, b_n]$.

Take f' ~~to~~ on I_n to ~~be~~ have the following graph.



Let $\chi(x) = 1 - |1-2x|$.



Then $f'(x) = 1 + c_n \chi\left(\frac{x-a_n}{b_n-a_n}\right)$.

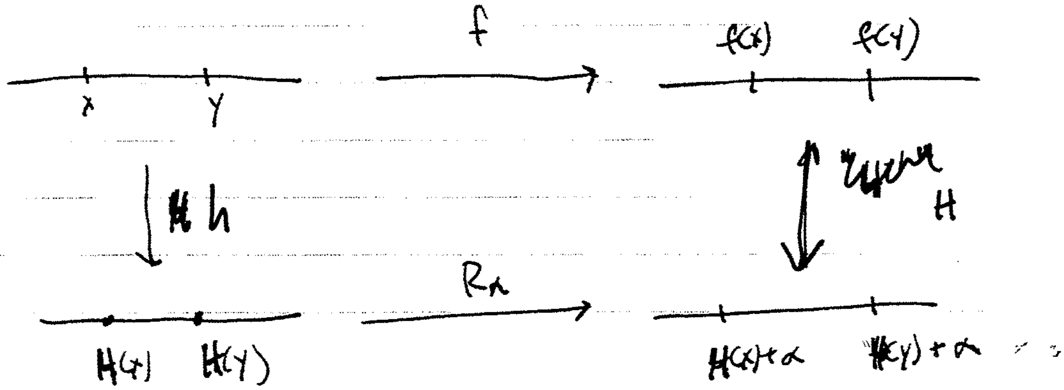
Since f maps $[a_n, b_n]$ to $[a_{n+1}, b_{n+1}]$

we need $\displaystyle\int_{a_n}^{b_n} f'(x)\,dx = \underbrace{(b_n - a_n)\left(1 + \frac{c_n}{2}\right)}_{\text{from the picture}} = b_{n+1} - a_{n+1}$

This gives $l_n\left(1+\frac{c_n}{2}\right) = l_{n+1}$ or $c_n = 2\left(\frac{l_{n+1}}{l_n} - 1\right)$.

Differentiability argument starts in lecture 9 page 5.

Why is f differentiable with a continuous derivative?



If $x \in \cup I_n$ this is true by construction.

Assume $x, y \notin \cup I_n$.

Want to show that $f'(x) = 1$ or

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = 1 \quad \text{or} \quad \lim_{y \to x} \frac{f(y) - f(x) - (y - x)}{y - x} = 0.$$

Assume $x < y$

Now $x = H(x) + \sum_{i : x_i \in [0, h(x)]} l_i$ $\qquad y = H(y) + \sum_{i : x_i \in [0, h(y)]} l_i$

$$y - x = H(y) - H(x) + \sum_{i : x_i \in (h(x), h(y)]} l_i$$

$$f(x) = H(f(x)) + \sum_{i:\, x_i \in [0,\, H(f(x))]} l_i$$

But $H(f(x)) = R_\alpha(H(x)) = H(x)+\alpha$

$$f(x) = H(x) + \alpha + \sum_{i:\, x_i \in [0,\, H(x)+\alpha]} l_i$$

$\mathring{l}_i$

$$f(y) - f(x) = (H(y)+\alpha) - (H(x)+\alpha) + \sum_{i:\, x_i \in [H(x)+\alpha,\, H(y)+\alpha]} l_i$$

$$= H(y) - H(x) + \sum_{i:\, x_i \in [H(x),\, H(y)]} l_{i+1}$$

$$f(y) - f(x) = (H(y) + \alpha) - (H(x) + \alpha) + \sum_{i:\, x_i \in [h(x)+\alpha,\, h(y)+\alpha]} l_i$$

$$= H(y) - H(x) + \sum_{i:\, x_i \in [h(x),\, h(y)]} l_{i+1}$$

so
$$\frac{f(y) - f(x) - (y-x)}{y-x} = \frac{\sum l_{i+1} - \sum l_i}{\underbrace{H(y) - H(x)}_{\text{positive}} + \sum l_i} \quad x_i \in (h(x), h(y))$$

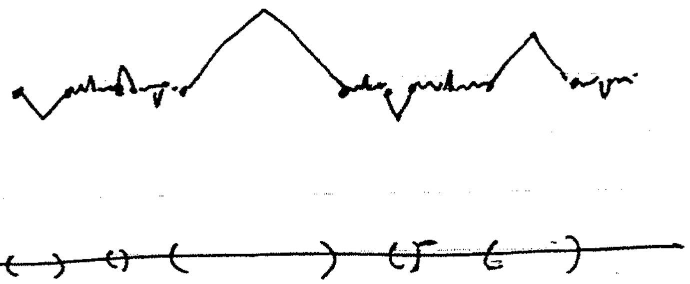
$$\le \frac{M \cdot \sum l_i - \sum l_i}{\sum l_i}$$

where $M = \max\limits_{i:\, x_i \in [H(x), H(y)]} \frac{l_{i+1}}{l_i}$ $\qquad \le M - 1.$

Now as $y \to x$, $H(y) \to H(x)$ and $M \to 1$. since $\frac{l_{i+1}}{l_i} > \varepsilon$ for only finitely many indices

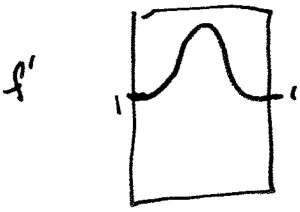
To get the lower bound argue with $\min \frac{l_{i+1}}{l_i}$.

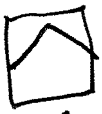
Picture of f'



For any $\varepsilon > 0$ only finitely many peaks are above $1+\varepsilon$ or below $1-\varepsilon$.
Thus f' is continuous.
At any $x \notin \bigcup I_n$ we have heights of wedges converging to 0 but the sum of the heights of all wedges is infinite.

Denjoy's example has a pretty wild derivative. Is it possible that this example could be made C^2 rather than just C^1? We could replace our maps from interval to interval by something smoother, but we will still have the structure of "peaks and valleys". Not so clear how to deal with this



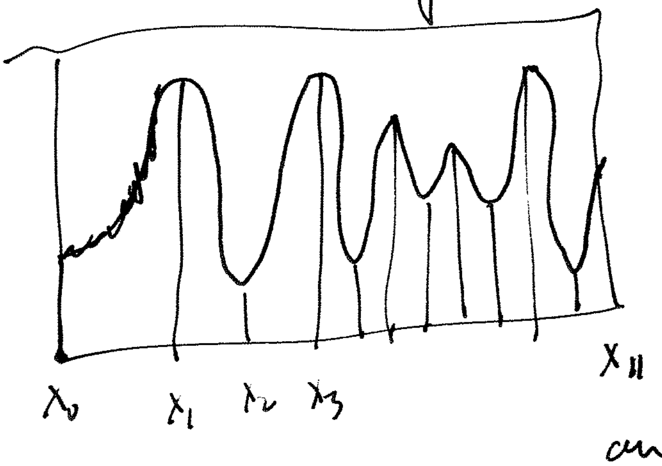


Denjoy shows that a C^2 diffeo. of the circle with irrational rotation number must be minimal. We will give his proof. We start with a definition.

Definition. We define the variation of a continuous function $g : [0,1] \to \mathbb{R}$ to be

$$\mathrm{var}(g) = \sup \left\{ \sum_{i=0}^{n-1} |g(x_{i+1}) - g(x_i)| : 0 = x_0 < x_1 < \dots x_n = 1 \right\}$$

where n can be arbitrary



We are summing the sizes of the jumps. When n is large we are looking at the behavior of smaller jumps. Roughness on a smaller scale.

i

Theorem (Denjoy) Let f be a circle homeomorphism with irrational rotation number ρ. If f is C^1 and $\log|f'|$ has bounded variation then f is topologically conjugate to R_ρ.

Remarks. You will show that if f is C^2 then $\log|f'|$ has bounded variation. Thus this result says there are no C^2 Denjoy counterexamples.

If f is not minimal then this reflected in the growth of the distortion of f^n as measured by the ~~[illegible]~~ size of ~~the~~ $(f^n)'$.

Lemma. Let f be a C^1 diffeo of the circle with irrational rotation number. If f is not minimal then ~~there is~~ there is an x_0 so that (depending on n)

$$|(f^n)'(x_0)| \cdot |(f^{-n})'(x_0)| \to 0 \text{ as } n \to \infty.$$

Proof. If f is not minimal there is an interval $I_0 \subset \mathbb{R}/\mathbb{Z}$ mapped to a point by h. ~~assume that~~ Let $I_j = f^j(I_0)$. These intervals are disjoint (since they map to different pts on an orbit under R_ρ)

Since $\sum |I_n| \leq 1$ we have $|I_n| \to 0$ as $n \to \pm\infty$.

$$\frac{|I_n| + |I_{-n}|}{2} = \frac{1}{2}\left(\int_{I_0} (f^n)'(z)\,dz + \int_{I_0} (f^{-n})'(z)\,dz \right)$$

$$= \int_{I_0} \frac{(f^n)'(z) + (f^{-n})'(z)}{2}\,dz$$

$$\geq \int_{I_0} \sqrt{|(f^n)'(z)(f^{-n})'(z)|}\,dz$$

$$\geq \sqrt{m} \cdot |I_0|$$

where $m = \min_{I_0} (f^n)'(z)(f^{-n})'(z)$. ~~Since~~ Pick x realizing the minimum then $m \to 0$ as $n \to \infty$.

Let's try to understand this result.

By the ~~[illegible]~~ rule for differentiating an inverse function

$$(f^{-n})'(x_0) = \frac{1}{(f^n)'(f^{-n}(x_0))} = \frac{1}{(f^n)'(x_{-n})}$$

So we have

$$\frac{|(f^n)'(x_0)|}{|(f^n)'(x_{-n})|} \longrightarrow 0 \quad \text{as } n \to \infty.$$

~~Taking logs turns the chain rule from a~~

Can analyze $(f^n)'(x_0)$ in terms of the chain rule.

$$(f^n)'(x_0) = \cancel{\prod_{i=0}^{n-1}} \; f'(x_0)\cdot f'(x_1) \cdots f'(x_{n-1}) = \prod_{i=0}^{n-1} f'(x_i).$$

Taking logs makes this a sum along an orbit:

$$\log (f^n)'(x_0) = \sum_{i=0}^{n-1} \log f'(x_i).$$

We have $\log \dfrac{|(f^n)'(x_0)|}{|(f^n)'(x_{-n})|} \to -\infty$ or

$$\sum_{i=0}^{n-1} \log f'(x_i) - \sum_{i=0}^{n-1} \log f'(x_{i-n}) \longrightarrow -\infty.$$

$$= \sum_{i=0}^{n-1} \log f'(x_i) - \log f'(x_{i-n}) \to 0$$

~~Usefulness~~ ~~At~~ Approaching this naively not surprising that we have an orbit ~~along~~ segment along which f ~~grows~~ is larger and one along which f ~~decreas~~ is smaller. The trickiness in the lemma allows us to locate these two points as x_0 and x_n.

x_0 . . . . x_n

x_n . . . . . x_0

Lemma. Let f be a circle homeomorphism with an irrational rotation number. Then there is a sequence $n_k \to \infty$ such that for any $x \in \mathbb{R}/\mathbb{Z}$ the intervals (x_i, x_{i+n_k}) (where $x_i = f^i(x)$) for $0 \le i \le n_k$ are disjoint.

Proof. Let R_ρ be the rotation with same rotation number as f. Let $y_k = R^k_\rho(x)$. for some x.

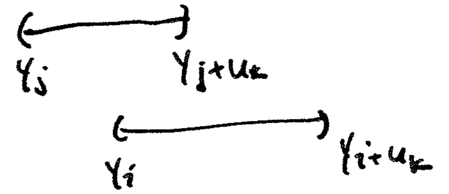
Let $n_0 = 1$ and define n_k recursively by

$$n_k = \min \{ i \in \mathbb{N} : \operatorname{dist}(y_0, y_i) < \operatorname{dist}(y_0, y_{n_{k-1}}) \},$$

$R^{n_k}(x)$ gives the sequence of closest approaches to 0. Since $R^{n_k}(x)$ is close to x. $|n_k \rho - m_k|$ is small so $\rho - \frac{m_k}{n_k}$ is very small. Gives "best rational" approximations to ρ.

Say intervals (y_i, y_{i+n_k}) and (y_j, y_{j+n_k}) overlap with $i, j \le n_k$

~~Say~~



Say $i > j$ then

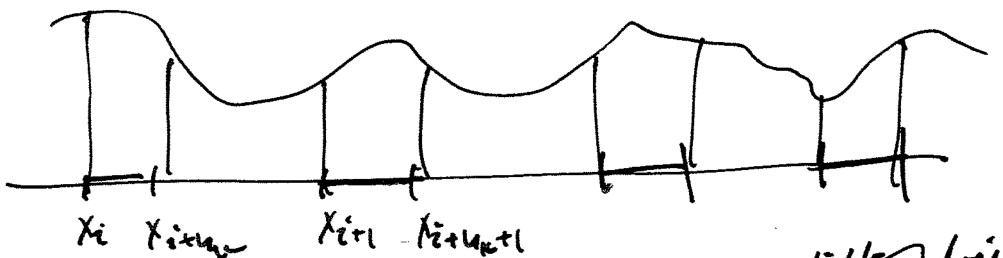
$$\begin{array}{ccc} \operatorname{dist}(y_j, y_i) & < & \operatorname{dist}(y_i, y_{i+n_k}) \\ \| & & \| \\ \operatorname{dist}(y_0, y_{i-j}) & & \operatorname{dist}(y_0, y_{n_k}) \end{array}$$

This violates the assumption that n_k is a closest return. (Holds if $|i-j| < n_{k+1}$).

Now let f be any homeomorphism with rotation number ρ. We have a semiconjugacy ~~H~~ h from f to R_ρ. The monotonicity of H implies that $f^i(x) \in [f^j(x), f^k(x)]$ if and only if

$$R_\rho^i(h(x)) \in [R_\rho^j(h(x)), R_\rho^k(h(x))].$$

We will use this to compare averages of functions at nearby points. ~~Any~~ g is continuous then $\left|\sum_{i=0}^{n_k} g(x_i) - \sum_{i=0}^{n_k} g(x_{i+n_k})\right| \le Var(g)$ since

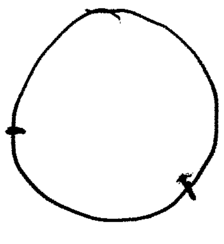


~~shift indices~~ take $y = x-n$

Apply this to $g = \log|f'|$ and we get

$$\left|\sum_{i=0}^{n_k} \log\{f'(x_i)\} - \log f'(x_{i-n})\right| \le var(\log|f'|)$$

This ~~contradicts~~ shows that f must be minimal



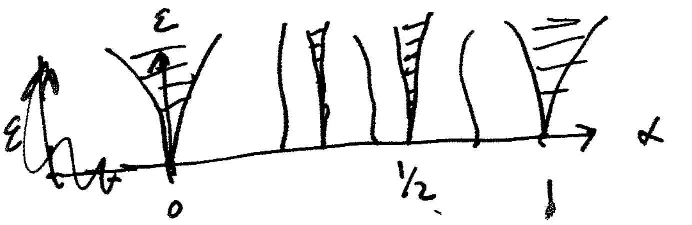
What happens in a generic family of homeomorphisms of the circle?

$\boxed{\text{Weakly coupled oscillators. } \alpha \text{ is ratio of periods. } \varepsilon \text{ is strength of coupling.}}$

Say we have a family like $f(x) = x + \alpha + \varepsilon \sin(2\pi x)$.

α corresponds to a rotation parameter and ε represents the "coupling" or "non-linearity". ~~This~~ We can plot the rotation number in the (ε, α) plane.

Arnold tongues.



Baire category:

large in a top. sense: generic – contains a dense open set; comeagre – contains a countable intersection of dense open sets.

If we increase α and fix ε, ρ is monotone increasing but not strictly monotone. At each irrational value f is conjugate to a rotation and ρ is strictly increasing as a function of α. So we have lines of irrational rotation #

Out of the ~~rat~~ rational points however we get "tongues". Each of these tongues gives a plateau. The result is a "Devil's staircase". In fact ρ is constant on a dense set. ρ is non-~~const~~ constant on a set of positive measure.

Phase locking is generic but not of full measure. Thus non-linear case is substantially different from the linear case.

Points in the interior of an Arnold tongue are top. conjugate to one another.

Def. f is structurally stable if ~~a~~ C^1 ~~when~~ all sufficiently C^1 close maps are topologically conjugate.

If $\rho(f)$ is irrational then f is not structurally stable.

~~A~~ Structural stability is dense.

Rings of Saturn show a similar pattern of gaps.