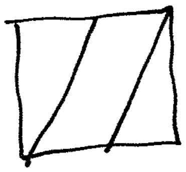
Next topic: Expanding maps of the circle.

Definition. A continuously differentiable map $f: \mathbb{R}/\mathbb{Z} \circlearrowleft$ is ~~called exp~~ <u>expanding</u> if $|f'(x)| > 1$ for all $x \in \mathbb{R}/\mathbb{Z}$.

Example: $f(x) = 2x \bmod 1$.



This is topologically equivalent to the map $g(z) = z^2$ on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

The conjugacy is given by $h(x) = e^{2\pi i x}$.

since $hf(x) = e^{2\pi i 2x}$
$gh(x) = (e^{2\pi i x})^2$

Note $h(x+1) = h(x)$, h is well defined on $\mathbb{R}/\mathbb{Z}$.

We begin with some topological properties of expanding maps.

Topology of expanding maps

① Prop. $\deg f\circ g = \deg f \cdot \deg g$.

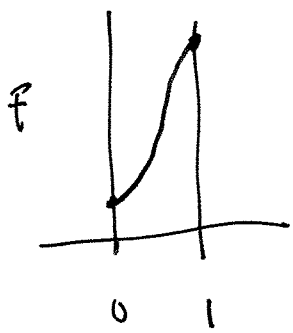
$$F(x+1) = F(x) + d \quad \text{or} \quad F\circ T = T^d \circ F$$

$$F\circ G T = F\circ T^{d_G} G = T^{d_G d_F} F\circ G.$$

> Cor. f^2 is or. preserving

② Prop. If f is expanding then $|\deg f| > 1$.

Proof.

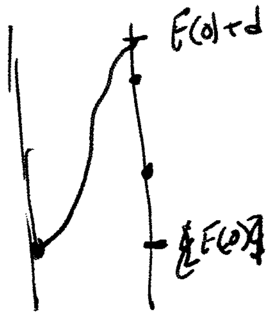


$$\frac{F(1)-F(0)}{1-0} = F'(x) \text{ for some } x \in (0,1)$$

$$\| \quad d.$$

F is strictly increasing or decreasing.

③ Prop. Every point has exactly d preimages.



Given $y \neq F(0) \mod \mathbb{Z}$ has exactly $x + \mathbb{Z} \cap [F(0), F(0)+d]$ contains d elements. $y_1 \dots y_d$

$x_j = F^{-1}(y_j)$ $j = 1 \dots d$ are distinct points and the $\pi(x_j)$ are all distinct. The points $\pi(x_j)$ are solutions of $f(x_j) = y$.

④ Prop. f has $|d-1|$ fixed points.

Proof. $f(x) - x$ has lift $F(x) - x$, so the degree is $(F(1) - 1) - (F(0) - 0) = F(1) - F(0) - 1 = d - 1$.

⑤ Cor. f^n has $|d^n - 1|$ fixed points!

⑥ Cor. f has ∞ many periodic points of ∞ many periods

Prop. Topologically conjugate maps have the same degree.

Proof Say $hf = gh$ or $f = h^{-1}gh$. Let G be a lift of g and H a lift of h. It follows that $F = H^{-1}GH$ is a lift of f. Say $\deg g = d$. The degree of f is d if $F \circ T = T^d F$.

We calculate assuming $\deg H = 1$.

$$F \circ T = H^{-1}GHT = H^{-1}GTH = H^{-1}T^dGH = T^dH^{-1}GH = T^dF$$

If $\deg H = -1$ we get

$$F \circ T = H^{-1}GHT = H^{-1}GT^{-1}H = H^{-1}T^{-d}GH = T^dH^{-1}GH = T^dF.$$

Let's look more carefully at the dynamics of the doubling map. ~~[illegible]~~ (Like rotation this is more "algebraic" than a general expanding map.) Let $f(x) = 2x \bmod 1$.

One way to see the ~~pro~~ dynamics of f is with binary expansions.

Let $x \in \mathbb{R}/\mathbb{Z}$. Write $x = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \dots$ where $a_i = 0$ or 1. Note that this expansion always exists but may not be unique: $\frac{1}{2} = \frac{1}{2} = 0 + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$

How does the doubling map act on binary expansions?

$$f(x) = a_1 + \frac{a_2}{2} + \frac{a_3}{2^2} + \frac{a_4}{2^3} + \dots \equiv \frac{a_2}{2} + \frac{a_3}{2^2} + \frac{a_4}{2^3} + \dots \mod$$

It drops the first term and shifts the others to the left.

$$(a_1, a_2, a_3 \dots) \to (a_2, a_3, a_4 \dots).$$

We can use this construction to construct periodic points since a periodic sequence gives rise to a periodic point.

$(0, 0, 1, 0, 0, 1, \dots)$

$(010010\dots$

$(100100100\dots$

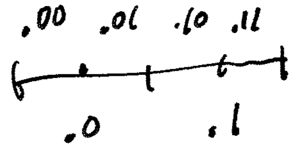
$001001\dots$

Prop. For the doubling map periodic points are dense.

Proof. Let $(a_1, a_2, a_3 \dots)$ ~~be~~ represent ~~some~~ an arbitrary point x. To find a periodic point within distance $\le 1/2^n$ of x consider:

$(a_1 a_2 a_3 \dots a_n a_1 a_2 a_3 \dots a_n \dots)$

Two binary expansions which agree for n terms correspond to ~~Prop The~~ points no more than $\frac{1}{2^n}$ apart.



(Note that there may be nearby points with different initial sequences.)

Prop. The doubling map has a dense orbit.

Proof. List all finite sequences of 0's and 1's.

0, 1, 00, 01, ~~10 11~~ 10, 11, 000, 001, 010, 011, ...

Concatenate them:

$a_1 a_2 \dots = 0100011011000001010011\dots$

Let x be the corresponding point in the circle. ~~Given~~ Given an arbitrary point y

$y = b_1 b_2 b_3 b_4 \dots$ ~~$a_1 a_2 a_3 a_4 \dots$~~ and an $n > 0$ we find the sequence ~~$a_j \dots a_n$~~

$b_1 = a_{j+1}$

$b_2 = a_{j+2}$

$\vdots$

$b_n = a_{j+n}$

$d(y, f^j(x)) \le \frac{1}{2^n}$

Then ~~$f(y)$~~ $f^j(x)$ starts with $b_1 \dots b_n$ so

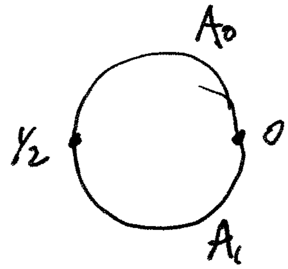
Many definitions have been proposed to capture the notion of "chaotic behaviour". One of these is the following:

f is <u>chaotic</u> if ~~that a dense poin~~ there is a point with a dense orbit and periodic points are dense.

Prop. The period doubling map is chaotic while circle homeomorphisms are not.

Proof. We have ~~al~~ proved the properties of the doubling map. If f is a circle homeomorphism with periodic points then all periodic points have the same period say p. Thus all periodic points are fixed points of f^p. ~~If pera~~ In particular the set of periodic points is closed. If it is also dense then it is everything and $f^p = id$. In particular there are no dense orbits.

What is going on geometrically?



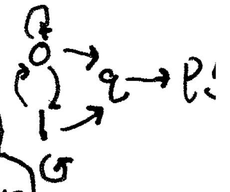
We have a decomposition of the circle into two pieces

$A_0 = [0, \frac{1}{2}]$ $A_1 = [\frac{1}{2}, 1]$.

$f(\partial A_0) = f(\partial A_1) \subset \partial A_0 \cup \partial A_1$.

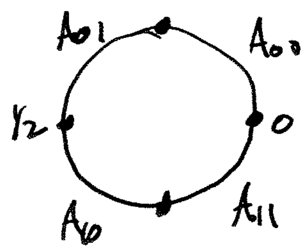
Given x we can construct an "itinerary" $w_0, w_1, w_2 \dots$ where $w_j = 0$ or 1 depending on whether $f^j(x) \in A_0$ or A_1.

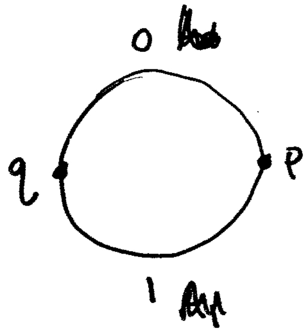
Not all transitions are possible for an itinerary. 0→1 1→0 0→q 1→q q→p p→p



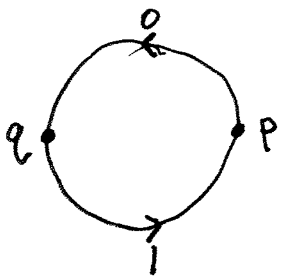
If we fix an n and consider trajectories of length n then we get a decomposition of the circle into intervals labelled by strings of n symbols.

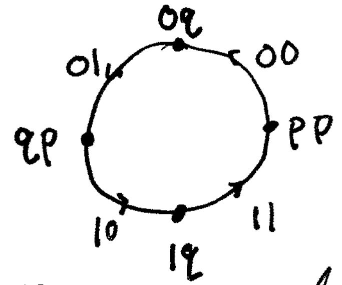
$n=2$

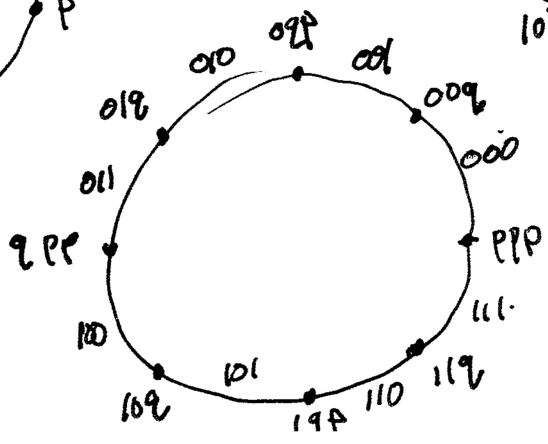




Markov prop
{p,q} taken into itself







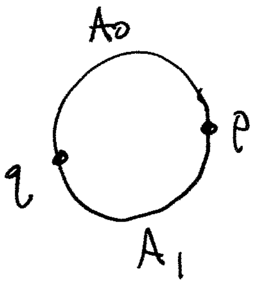
Label points by their itineraries of length n.

We now allow f to have non-constant derivative

Thm. Let $f: S \circlearrowleft$ be an expanding map from the circle to itself with degree 2. Let Σ_2 be the sequence space on the symbols 0 and 1. Let $\sigma: \Sigma_2 \circlearrowleft$ be the left shift. Then there is a semi-conjugacy $h: \Sigma_2 \to S$ so that $h\sigma = fh$.

Proof. f has a unique fixed point. Denote it by p. $f^{-1}(p)$ contains 2 points one of which is p. Let q be the second point.

Let $A_0 = (p, q)$ and $A_1 = (q, p)$. Thus $S = A_0 \cup A_1 \cup \{p, q\}$.



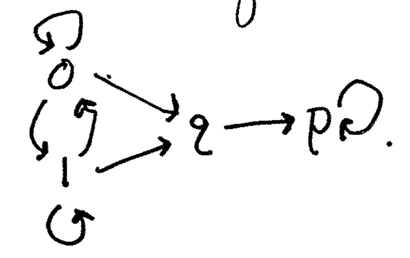
Fix an $n \geq 1$. Given $x \in S$ we can consider its trajectory of length n: $x, f(x), f^2(x) \dots f^{n-1}(x)$.

We associate to x a sequence of n-symbols $\omega_0 \dots \omega_{n-1}$ where:

$$\omega_j = \begin{cases} 0 & f^j(x) \in A_0 \\ 1 & \text{if } f^j(x) \in A_1 \\ p & f^j(x) = p \\ q & f^j(x) = q \end{cases}$$

S decomposes into subsets with different itineraries $\omega_0 \dots \omega_{n-1}$. We want to understand this decomposition.

Note that itineraries follow the transitions given by:



A vertex of level n corresponds to a word $\omega_0 \dots \omega_{n-1}$ which contains a p or q.

For a word $w_0 \ldots w_{n-1}$ containing only 0's and 1's we write $A_{w_0 \ldots w_{n-1}}$ for the set of x with n-step itinerary $w_0 \ldots w_{n-1}$.

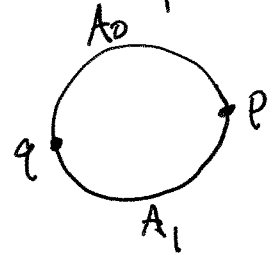
Claim. ① For any $w_0 \ldots w_{n-1} \in \Sigma_2$ $A_{w_0 \ldots w_{n-1}}$ is a non-empty open interval.

② The closure $\bar{A}_{w_0 \ldots w_{n-1}}$ is a closed interval with boundary consisting of vertices of level n.

③ $f^n | A_{w_0 \ldots w_{n-1}}$ is a homeomorphism onto $S-P$. $f^{n-1} | \bar{A}_{w_0 \ldots w_{n-1}}$ is a homeomorphism onto $\bar{A}_{w_{n-1}}$.

We prove the claim by induction.

For $n=1$ this is a statement about the decomposition and is easily seen to hold.



Assume the statement for n and we will prove it for $n+1$.

Let $w_0 \ldots w_n$ be a word of 0's and 1's of length $n+1$.

The definition of itinerary shows that $A_{w_0 \ldots w_n}$ is a sb subset of $A_{w_0 \ldots w_{n-1}}$.

It also shows that $A_{w_0 \ldots w_n}$ consists of those points x in $A_{w_0 \ldots w_{n-1}}$ for which $f^n(x) \in A_{w_n}$. By induction $f^n | A_{w_0 \ldots w_{n-1}}$ is a homeomorphism to $S - p$. Let $g = f^n | A_{w_0 \ldots w_{n-1}}$. $g^{-1}(A_{w_n})$ is an interval since A_{w_n} is an interval

In fact $f^n|\bar{A}_{w_0 \ldots w_n}$ is a homeomorphism and the endpoints of $\bar{A}_{w_0}$ which are p and q. $g^{-1}(p)$ and $g^{-1}(q)$ are vertices of level $n+1$.

(3) Need to check that $f^{n+1}|A_{w_0 \ldots w_n}$ is a homeomorphism onto $S-p$. $f^{n+1}|A_{w_0 \ldots w_n}$ is the composition of two homeomorphisms

$$A_{w_0 \ldots w_n} \xrightarrow{f^n} A_{w_n} \xrightarrow{f} S-p.$$

Length estimate for intervals $A_{ab\ldots w_{n-1}}$.

Recall that f' is continuous and $|f'(x)| > 1$

Since the circle is compact this implies

$|f'(x)| \geq |f'(x_0)| = K > 0$. So we have a uniform

estimate. Now $f^n : A_{ab\ldots w_{n-1}} \to [p, p+1]$ is 1-1

so $$\int_{A_{ab\ldots w_{n-1}}} (f^n)'(x)\,dx = (p+1) - p = 1$$

But $$1 = \int_A (f^n)'(x)\,dx = \int_A f'(x) \cdot f'(f(x)) \cdot \ldots f'(f^{n-1}(x))\,dx$$

$$\geq \int_A K^n\,dx \,.$$

$$= K^n \cdot |A|$$

which gives $|A_\omega| \leq \frac{1}{K^n}$.

Now we construct the semi-conjugacy from Σ_2 to $S = \mathbb{R}/\mathbb{Z}$.

Let $\omega = \omega_0 \omega_1 \ldots$ be an infinite squence.

Consider $\bigcap_{n=0}^{\infty} \bar{A}_{\omega_0 \ldots \omega_n}$.

These intervals are nested and decreasing in length to zero, so the intersection is a single point. Call it $h(\omega)$.

h is continuous. Given $\varepsilon > 0$ choose an n so that $\frac{1}{K^n} \le \varepsilon$. Let $\delta = \frac{1}{2^n}$. If $d(\omega, \omega') \le \frac{1}{2^n} = \delta$ then $\omega_0 \ldots \omega_{n-1} = \omega_0' \ldots \omega_{n-1}'$ so $h(\omega)$ and $h(\omega')$ are in $A_{\omega_0 \ldots \omega_{n-1}}$ which has diameter $\le \frac{1}{K^n} = \varepsilon$.

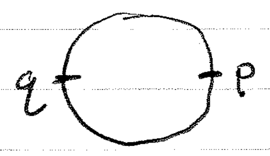
h is a semi-conjugacy. | Holder continuity but no better.

$f \circ h = h \circ \sigma$.

$$f(h(\omega)) = f \bigcap_{n=0}^{\infty} \bar{A}_{\omega_0 \ldots \omega_n} = \bigcap_{n=0}^{\infty} f(\bar{A}_{\omega_0 \ldots \omega_n})$$
$$= \bigcap_{n=0}^{\infty} \bar{A}_{\omega_1 \ldots \omega_n}$$
$$= h(\sigma(\omega)).$$

~~When does h map two po~~ ω, ω'

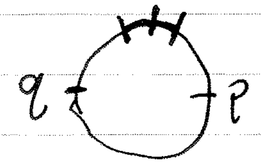
When do two points ~~map to~~ have the same image under h? $h(\omega)=h(\omega')$?



~~Either $\omega_0 = \omega_0'$~~

This means that for each finite sequence $\omega_0 \dots \omega_{n-1}$, $\omega_0' \dots \omega_{n-1}'$ the sets $A_{\omega_0 \dots \omega_{n-1}}$ and $A_{\omega_0' \dots \omega_{n-1}'}$ intersect.

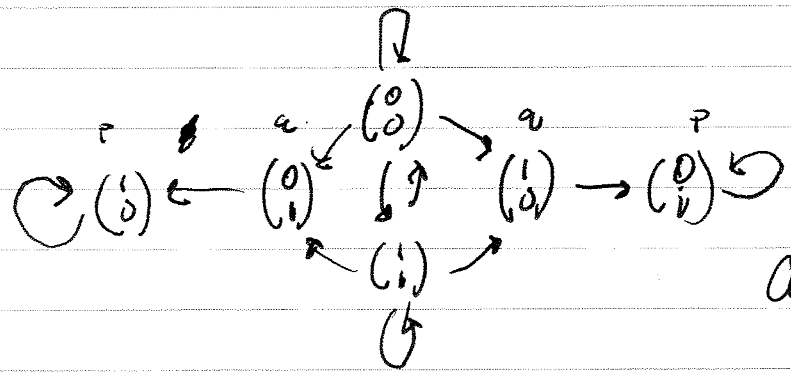
There are 4 cases $\omega_0 = \omega_0' = 0$ or $\omega_0 = \omega_0' = 1$



or $\omega_0 \neq \omega_0'$ and $A_{\omega_0} \cap A_{\omega_0'} = q$

or $\omega_0 \neq \omega_0'$ and $\omega_0 \cap \omega_0' = p$.

~~If A_{ω_0}~~



Allowable transition

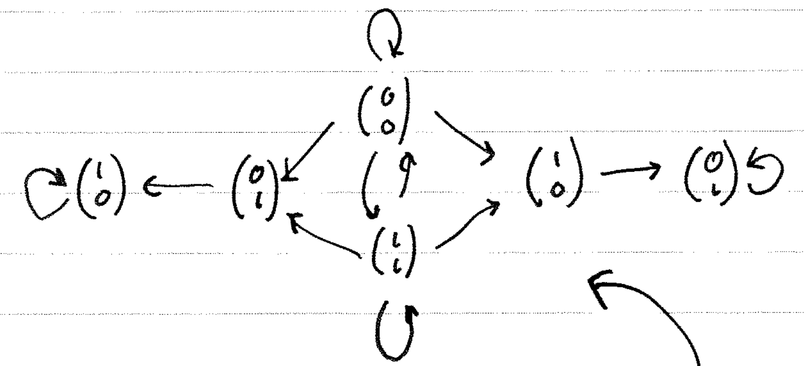
(1) → (2) → (3)

Going from (2) to (3) the symbols switch.

We can say

Prop.

Two sequences $\omega = \omega_0\omega_1\ldots$ and $\omega' = \omega_0'\omega_1'\ldots$ have the same image under h if the sequence of pairs $\binom{\omega_0}{\omega_0'}\binom{\omega_1}{\omega_1'}\cdots$ is compatible with the graph:



Example.

$$\begin{matrix}0110000\ldots\\0101111\ldots\end{matrix}$$

$$\binom{0}{0}\binom{1}{1}\binom{1}{0}\binom{0}{1}\binom{0}{1}\cdots$$

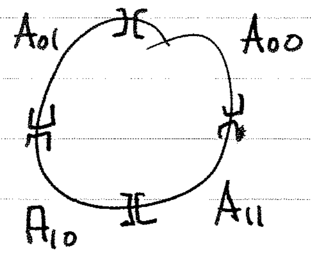
or

$$\begin{matrix}0000\ldots\\1111\ldots\end{matrix}$$

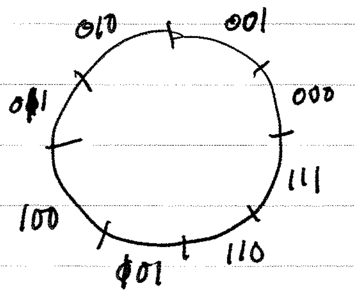
$$\binom{0}{1}\binom{0}{1}\binom{0}{1}\binom{0}{1}\cdots$$

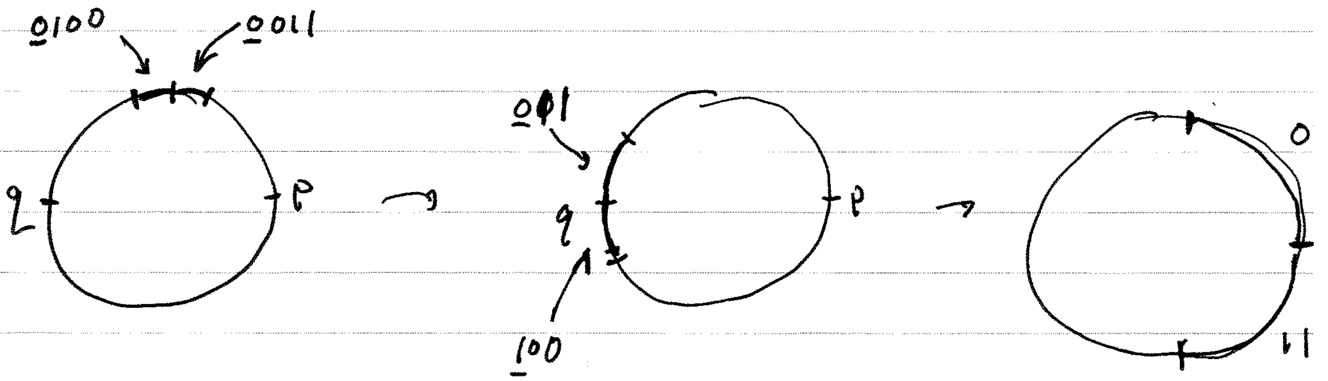
This graph also determines ~~the~~ when 2 binary expansions represent the same number.

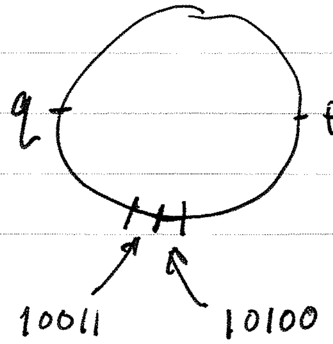
Recall that when we were coding intervals we described the ~~codes~~ itineraries of the interior points.



~~When~~ The coding of the boundary points is ambiguous. Instead of coding the boundary points we want to compare the codes of the two intervals on either side.







Prove that f_m and f_n are only top. conj. if $n = m$, $n \in \mathbb{Z} - \{0, \pm 1\}$

Thm. If f and g are expanding maps of degree $+2$ then f and g are topologically conjugate.

$$\Sigma_2$$
$$\swarrow \quad \searrow$$
$$\mathbb{R}/\mathbb{Z} \dashrightarrow \mathbb{R}/\mathbb{Z}$$

Quotient topology?

Very different from circle homeomorphisms

Thm. If f is an expanding map then periodic points are dense, f is top. transitive.