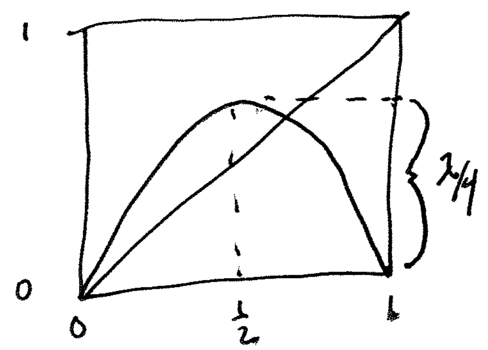
The next topic we want to consider is interval maps.

Example: $f_\lambda(x) = \lambda x(1-x)$.

$f_\lambda : [0,1] \longrightarrow [0,1]$ for $0 \le \lambda \le 4$.



We have seen examples of non-chaotic behaviour and chaotic behaviour. The logistic family incorporates both. Recall that the logistic equation models levels of population of an insect population after n generations. It came as a surprise to biologists in the 1970's that this simple model could produce chaotic behaviour.

~~Say (X, f) is a dyn dynamical system.~~ Let $f: X \to X$ be a

Definition. The limit set of a point x is defined to be $\Lambda^*(x) = \bigcap_{n=0}^{\infty} \overline{\{ f^i(x) : i \geq n \}}$.

$\Lambda^*(x) \subset \overline{O^+(x)}$ but $\Lambda(x)$ might be smaller.

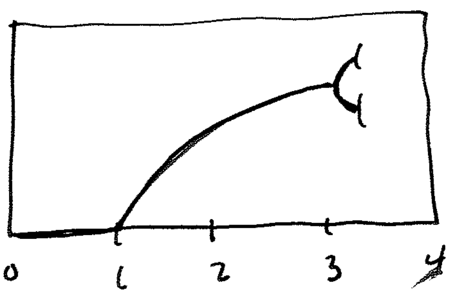
$y \in \Lambda(x)$ if there is a sequence $n_j \to \infty$ so that $f^{n_j}(x) \to y$.

Recall that a periodic point x of period n is attracting if for y suff. close to x $d(f^j(y), f^j(x)) \to 0$.
If $|(f^n)'(x)| < 1$ then x is attracting. If $|(f^n)'(x)| > 1$ then x is not attracting.

~~Def~~ We say that a bifurcation is a parameter λ for which the dynamics of f_λ changes.

We construct the following picture by plotting points in the x, λ plane. We call this a bifurcation diagram. It should be thought of as plotting $\Lambda(x)$ ~~as a fcn~~ for the map f_λ as a function of λ. It is a fact that for these maps $\Lambda(x)$ is typically independent of the choice of x.

If we extend the diagram it looks like



The picture appears to start with a sequence of periodic points of period 1, 2, 4, 8...

We can confirm this algebraically up to a point.

Fixed points of the equation are solutions of $f(x)=x$ or roots of $g(x)=f(x)-x$.

The implicit function theorem says that if x_{λ_0} is a root of g_{λ_0} then we have ~~nearby roots~~ roots x_λ for λ near λ_0 if $g'(x_{\lambda_0})\neq 0$.

This means $f'(x_\lambda)\neq 1$. We get a potential bifurcation when we have a simultaneous solution of $x=\lambda x(1-x)$ and $1=\lambda(1-2x)$. [Note $f'(0)=\lambda$. 0 stops being attracting.]

This occurs only at $\lambda=1$ $x=0$. What happens here is that a second fixed point $p=1-\frac{1}{\lambda}$ is created with $f'(p)=2-\lambda$. At $\lambda=1$ p starts being attractive. ~~p stops being~~ Note that it is the <u>attractive</u> periodic points that show up in the bifurcation diagram.

Since $f_\lambda'(p) = 2-\lambda$, p stops being attractive at $\lambda=3$ (where $f_\lambda'(p) = -1$). Apparently a circle of period 2 is created. The bifurcation equations for creation of points of period 2 are $f_\lambda^2(x_\lambda) - x_\lambda = 0$ and ~~$(f_\lambda^2)'(x_\lambda) - 1 = 0$~~ or $(f_\lambda^2)'(x_\lambda) = 1$. Since $(f_\lambda)'(x_\lambda) = 1$ these conditions hold. To find points of period 2 we solve the 4-th degree equation $f_\lambda^2(x) - x = 0$ but we already know two of the roots so it suffices to solve the quadratic equation $\frac{f^2(x)-x}{f(x)-x}$.
The roots are $p_\pm = \frac{1+\lambda \pm \sqrt{(\lambda+1)(\lambda-3)}}{2\lambda}$ for $\lambda \geq 3$.

$(f_\lambda^2)'(p_+) = f_\lambda'(p_+) f_\lambda'(p_-) = 4 + 2\lambda - \lambda^2$. This becomes less than 1 in abs. value for $3 < \lambda < \cancel{3} 1+\sqrt{6}$.

As λ increases up to some particular value λ_0 the map creates ~~more~~ additional points of period 2^n but each f_λ has only finitely many periodic points. For $\lambda > \lambda_0$ f_λ has periodic points of ∞ many periods.

We will build a topological model of of this period doubling phenomenon.

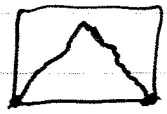
We say that a map $g: [0,1] \to [0,1]$ is ~~unimodal if~~ symmetric if $g(x) = g(1-x)$.
We say a symmetric map is unimodal if it is increasing on $[0,\frac{1}{2}]$ and decreasing on $[\frac{1}{2},1]$

Proposition. Let f be a symmetric unimodal map with periodic points of periods $p_1, p_2, p_3 \ldots$ ~~Then~~ and no others. Then there is a symmetric unimodal map g with periodic points of periods $1, 2p_1, 2p_2 \ldots$ and no others.

Cor. For any n there is a symmetric unimodal map with periodic points of period $1, 2, 4, 8 \ldots 2^n$ and no others.

$: [0,1] \circlearrowleft$

f is unimodal and symmetric if $f(x) = f(1-x)$,
f is monotone increasing on $[0, 1/2]$ and monotone
decreasing on $[1/2, 1]$.
$f(0) = f(1) = 0$.



Periods of periodic points. x has period n if

$f^n(x) = x$ but $f^m(x) \neq x$ for $0 < m < n$.

(sometimes called "least period" or "prime period".)

What effect does the unimodal condition have on the collections of possible periods of periodic points?

Proposition. Let f be a symmetric unimodal map with periodic points of periods exactly $p_1, p_2 \dots$ Then there is a symmetric unimodal map with periodic points of periods exactly $1, 2p_1, 2p_2 \dots$

Remarks. Every map of the interval has points of period 1. ($f(0) = 0$)

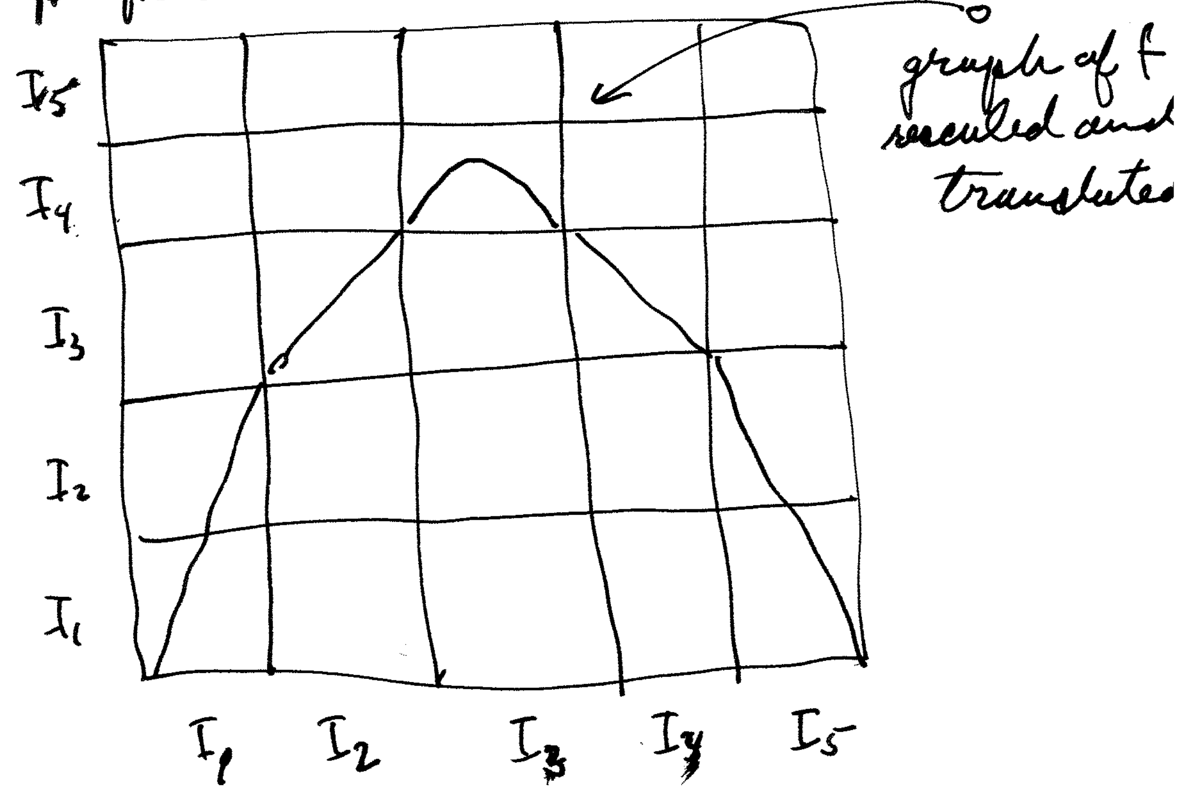
The list of periods $p_1, p_2 \dots$ can be finite or infinite.

Let f be unimodal, symmetric and satisfy

$f(0) = f(1) = 0$.

Using f we construct a new function g with the same properties.

Note



Let $L_i(x) = \frac{x+i-1}{5}$ $\quad L_j : [0,1] \to I_j$ is affine.

Let $S(x) = 1-x$. Note $f(S(x)) = f(x)$ since f is symmetric.

Def. of g.

$$g|I_1(x) = 2x$$

$$g|I_2 = L_3 \circ L_2^{-1}$$

$$g|I_3 = L_4 \circ f \circ L_3^{-1}$$

$$g|I_4 = L_3 \circ S \circ L_4^{-1}$$

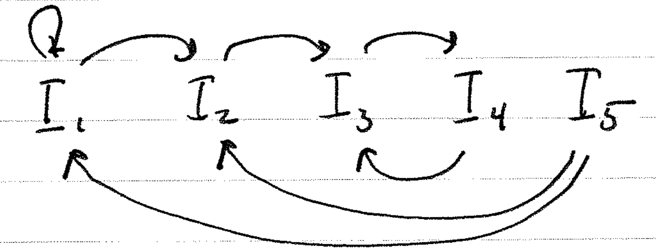
$$g|I_5(x) = (1-2x).$$

Proof. We will locate the periodic points of g.

Where do we see fixed points for g?

$p = 0$ and $q = 0.6$

How do points move?



Where are other periodic points located?

There are no periodic points in I_5.

There are no periodic points in I_1 other than p,
since any periodic point in I_1 remains in I_1 for all time.
Any point other than p leaves I_1 eventually
and never returns.

There are no periodic points in I_2.

Every periodic point in $I_3 \cup I_4$ other than q
moves from I_3 to I_4 and from I_4 to I_3.

In particular if $f^n(x) = x$ then n is even.

$g|I_3 = L_4 \circ f \circ L_3^{-1}$ $\qquad g|I_4 = L_3 \circ S \circ L_4^{-1}$

$g^2|I_3 = L_3 \circ S \circ L_4^{-1} \circ L_4 \circ f \circ L_3^{-1} = L_3 \circ S \circ f \circ L_3^{-1}$

$g^2|I_4 = L_4 \circ f \circ L_3^{-1} \circ L_3 \circ S \circ L_4^{-1} = L_4 \circ f \circ S \circ L_4^{-1}$.

$g^{2n}|I_4 = L_4 \circ (f \circ S)^n \circ L_4^{-1}$

~~If $g^{2n}(x) = x$ (we can assume~~

$= L_4 \circ f^n \circ L_4^{-1}$

If $f^n(x) = x$ then $g^{2n}(L_4(x)) = L_4 \circ f^n \circ L_4^{-1} \circ L_4(x) = L_4 f^n(x) = \ldots$

($f^m(x) \neq x$ for $1 < m < n$)

If $g^m(L_4(x)) = L_4(x)$ then (i) m is even and $f^{m/2}(x) = x$.

So a ~~point of period~~ periodic ~~n~~ n for f has period $2n$ for g.

Conversely a periodic pt. ~~point of period~~ $2n$ for g has period n for f.

Corollary. There are symmetric unimodal maps with periods of periodic points equal to $1, 2, 4, 8 \ldots 2^n$ for any $n \geq 0$.

Corollary. There is a symmetric unimodal map with periods of periodic points equal to $1, 2, 4, 8, \ldots 2^n$ i.e. all powers of 2.

Thm. (Period 3 implies chaos.) If f is a continuous map of the interval (not assumed to be unimodal) with a point of period 3 then f has points of all periods.

Remark: p has period n if $f^n(p) = p$ but $f^m(p) \neq p$ for $m < n$.

~~We see from this diagram that the only possible period~~

~~Pe~~

We see from this diagram that periodic points must be contained in I_0 or $I_2 \cup I_3$.

The only periodic point in I_0 is 0. ~~The only periodic point in~~ Let $p = I_2 \cup I_3$. 0 and p are fixed points.

Let $q \neq p$ be a periodic point in $I_2 \cup I_3$. Say $g^n(q) = q$. Say $q \in I_3$ then $g(q) \in I_2$, $g^2(q) \in I_3$

$g^{2j}(q) \in I_3$, $g^{2j+1}(q) \in I_2$. If $g^n(q) = q$ then $g^n(q) \in I_3$. If n is odd then $g^n(q) = g^{2j+1}(q) \in I_2$ but this contradicts the assumption $q \notin I_2 \cap I_3$.

$$\begin{tikzcd}
I_3 \arrow[r, "L_2 \circ h \circ L_3^{-1}"] & I_2 \arrow[r] & I_3 \arrow[r] & I_2 \arrow[r] & I_3 \\
I \arrow[u, "L_3"] \arrow[r, "h"] & I \arrow[u, "L_2"] \arrow[r, "f"] & I \arrow[u, "L_3"] \arrow[r, "h"] & I \arrow[u] \arrow[r, "f"] & I \arrow[u]
\end{tikzcd}$$

~~Al~~ Say $n = 2m$. $g^n(q) = ~~(f \circ h)^m(q) \neq f^m(q)~~$

$L_3 \circ \underbrace{(f \circ h)^m}\circ L_3^{-1}$

$= L_3 \circ f^m \circ L_3^{-1}(q)$

So $L_3^{-1}(q)$ is a point of period m for f.

Conversely if y is a point of period m for f then $L_3(y)$ is a point of period $2m$ for g.

Cor. There is a unimodal symmetric map with periodic points ~~of all p~~ periods $1, 2, 4, 8 \ldots 2^k \ldots$

Prop. Every periodic point of g_∞ is repelling; of period 2^n. There is an invariant Cantor set C so that every point x which doesn't eventually hit a repelling periodic point satisfies $\Lambda(x) = C$.

~~Cor~~ C is a minimal set for g_∞.

~~Pre~~

Key claim from proof: $g^2|I_3$ is conjugate to f. Specifically $g^2|I_3 = L_3 \circ f \circ L_3^{-1}$

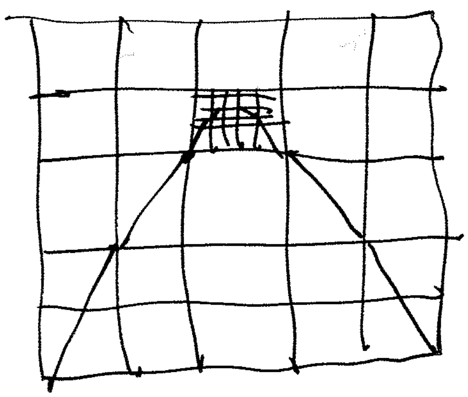
$$\begin{tikzcd} I_3 \arrow[r, "g|I_3"] & I_2 \arrow[r, "g|I_2"] & I_3 \\ I \arrow[u, "L_3"] \arrow[r, "h"] & I \arrow[u, "L_2"] \arrow[r, "f"] & I \arrow[u, "L_3"] \end{tikzcd}$$

$$\begin{tikzcd} I_3 \arrow[r, "g^2|I_3"] & I_3 \\ I \arrow[u, "L_3"] \arrow[r, "f"] & I \arrow[u, "L_3"] \end{tikzcd}$$

or

$L_3^{-1}(g^2|I_3)L_3 = f$

~~Letras~~ Given f let us call the resulting map g, R(f). We can think about iterating R. What does $R^n(f)$ look like?



Involves ~~making~~ inserting f in a box of size $(\frac{1}{3})^n \times (\frac{1}{3})^n = L_3^n(I) \times L_3^n(I$

~~Unaltered~~

The sequence of functions $R^n(f)$ converges uniformly to a limit which is independent of f. Call this limit g_∞. Then g_∞ has periodic points of period 2^n for all n.

We proved $g^2|I_3 = L_3 \circ f \circ L_3^{-1}$. For g_∞ we get $g_\infty^2|I_3 = L_3 \circ g_\infty$

Does g_∞ have any other interesting dynamics?

~~$I_2 \cup I_3$ is an invariant~~ $\qquad g_\infty^2 | I_3 = L_3 \circ g_\infty \circ L_3^{-1}$

g_∞ satisfies $g_\infty^2(I_3) = I_3$ and ~~$L_3 \circ g_\infty \circ L_3^{-1} = g_\infty^2$~~.

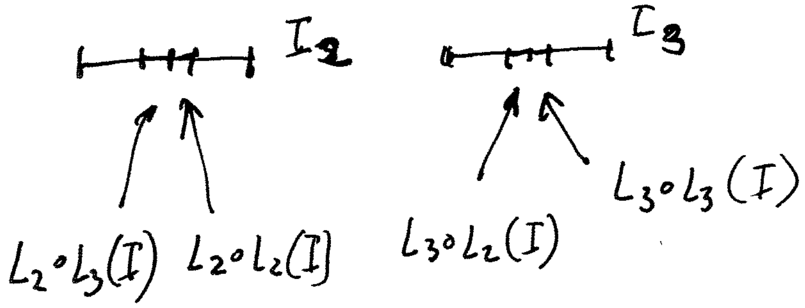
Conjugate to itself with a time change.

$$\begin{tikzcd} I_3 \arrow[r, bend left] & I_2 \arrow[l, bend left] \end{tikzcd}$$

$g_\infty(I_3) = I_2$.
$g_\infty(I_2) = I_3$.

$I_3 = L_3(I) \qquad I_2 = L_2(I)$

$L_3(I_3) = L_3^2(I)$ comes back to itself after 4 iterates



$$g^4(L_3 \circ L_3(I))$$

$$g^4 \,|\, L_3 \circ L_3(I) = g^2$$

$$\begin{tikzcd} I_3 \arrow[r, "g^2"] & I_3 \\ I \arrow[u] \arrow[r, "g"'] & I \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} L_3(I) \arrow[r, "g^4"] & L_3(I) \\ I \arrow[u, "L_3"'] \arrow[r, "g^2"] & I \arrow[u, "L_3"'] \end{tikzcd}$$

$$\begin{tikzcd} L_3(I) \arrow[r, "g^2"] & L_3(I) \\ I \arrow[u, "L_3"] \arrow[r, "g"] & I \arrow[u, "L_3"'] \end{tikzcd}$$

1975 1964

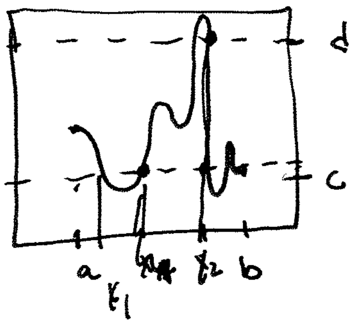
Thm. (Li-Yorke, Sarkovskii) Let f be a map from the interval to itself (not assumed to be unimodal). If f has a point of period 3 then f has periodic points of all periods.

(Period 3 ⇒ chaos)

Lemma. If $J_1, J_2 \subset I$ are intervals with $J_2 \subset f(J_1)$ then there is a closed interval $J_0 \subset J_1$ with

~~$f(J_0) = J_2$~~ $f(J_0) = J_2$.

Proof



Pick x_1 mapping to c and x_2 mapping to d.

Let x_2' be the point in $[x_1, x_2]$ closest to x_1 mapping to d.

By construction $f(x) \le d$ on $[x_1, x_2']$.

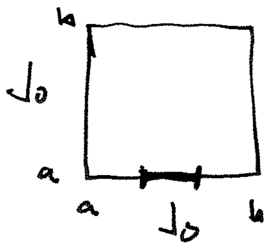
Let x_1' be the point in $[x_1, x_2']$ closest to x_2' ~~at which~~ mapping to c. By construction $f(x) \ge c$ on $[x_1', x_2']$. Furthermore $f([x_1', x_2'])$ contains $[c, d]$ by the IVT.

Cor. Suppose we have intervals $J_0, J_1 \dots J_n$ with $f(J_i) \supset J_{i+1}$ then there is an $I_n \subset J_0$ with $f^i(I_n) \subset J_i$ for $i = 0 \dots n$ and $f^n(I_n) = J_n$.

Proof. Apply the previous proposition repeatedly.
$f(J_0) \supset J_1$ so there is an $I_1 \subset J_0$ with $f(I_1) = J_1$
$f^2(I_1) \supset J_2$ so there is an $I_2 \subset I_1$ with $f^2(I_2) = J_2$.
~~Proof by induction on n.~~ Since $I_2 \subset I_1$ we have $f(I_2) \subset J_1$.
Proof by induction on n.

Lemma. Say we have a sequence $J_0, J_1, \ldots J_n = J_0$ with $f(J_i) \supset J_{i+1}$ then there is a periodic point ~~x within~~ $x \in J_0$ with $f^i(x) \in J_i$ and $i = 0 \ldots n$ and $f^n(x) = x$. (Since we don't assume intervals are disjoint the period might be smaller than n.)

Proof. By previous result there is an $I_n \subset J_0$ with $f^i(I_n) \subset J_i$ and $f^n(I_n) \supset J_0$.



Say $J_0 = [a, b]$. Say x_0 maps to $\underline{a}$ x_1 maps to

Consider $h(x) = f^n(x) - x$ on $[x_0, x_1]$ (or $[x_1, x_0]$).

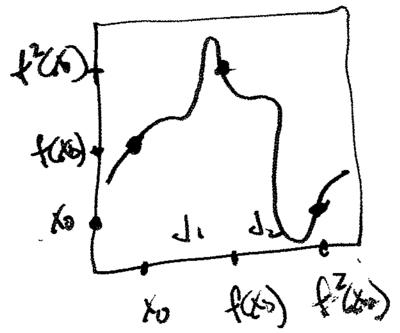
$h(x_0) = a - x_0 \geq 0$ $\quad h(x_1) = b - x_1 \leq 0$. By the MVT there is a point ~~in~~ $x \in [x_0, x_1] \subset J_0$ with $h(x) = 0$.

Proof of the theorem.

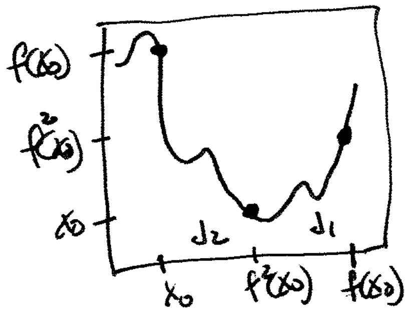
Consider a point of period 3. Let x_0 be the smallest point on the orbit. Either (1) $x_0 < f(x_0) < f^2(x_0)$

or (2) $x_0 < f^2(x_0) < f(x_0)$.

In case 1



Case 2



In case 1 let $J_1 = [x_0, f(x_0)]$, $J_2 = [f(x_0), f^2(x_0)]$

$f(\partial J_1) = \{f(x_0), f^2(x_0)\}$ so $f(J_1) \supset J_2$

$f(\partial J_2) = \{f^2(x_0), x_0\}$ so $f(J_2) \supset J_1$ and J_2.

$$\begin{tikzcd} J_1 \arrow[r, bend left] & J_2 \arrow[l, bend left] \arrow[loop right] \end{tikzcd}$$

In case 2 let $J_1 = [f^2(x_0), f(x_0)]$, $J_2 = [x_0, f^2(x_0)]$

$f(\partial J_1) = \{x_0, f^2(x_0)\}$ $f(\partial J_2) = \{f(x_0), x_0\}$

$$\begin{tikzcd} J_1 \arrow[r, bend left] & J_2 \arrow[l, bend left] \arrow[loop right] \end{tikzcd}$$

To create a periodic point of period $n \neq 3$ or 1 we choose a cycle compatible with the graph of f having length n which is not the result of repeating a cycle of shorter length.

Say $\underbrace{J_2 \to J_2 \to J_2 \cdots J_2}_{n-1} \to \underbrace{J_1}_{1}$

By the lemma we have an interval $I_n \subset J_2$ so that

$$f^j(I_n) \subset J_2 \text{ for } j = 0 \dots n-2$$
$$\subset J_1 \text{ for } j = n-1$$
$$= J_2 \text{ for } j = n.$$

By the previous lemma there is an $x \in I_n$ with $f^n(x) = x$. Need to show that $f^m(x) \neq x$ for $0 < m < n$.

Say $f^m(x) = x$ with $0 < m < n$.

~~Now $f^{n-m}(x) = f^{n-m} \circ f^m(x)$ so $f^{n-m}(x)$~~

Apply f^{n-m-1} to both sides.

$$f^{n-m-1} \circ f^m(x) = f^{n-m-1}(x) \in J_2$$
$$\|$$
$$f^{n-1}(x) \in J_1$$

So $f^{n-1}(x) \in J_1 \cap J_2$. But $J_1 \cap J_2$ intersect in the point of period 3

Coding for the point of period 3 is (case 1). ~~Cannot~~ ~~have 2~~ Can this be of the above form? Cannot have 2 successive J_1's

$$\begin{tikzcd} & J_1 \arrow[dr] & & & J_1 \arrow[dr] & \\ J_1 \arrow[ur] \arrow[dr] & & J_2 \arrow[r] & J_1 \arrow[ur] \arrow[dr] & & J_2 \\ & J_2 \arrow[ur] & & & J_2 \arrow[ur] & \end{tikzcd}$$

$J_1\, J_2\, J_2\, J_1\, J_2\, J_2$ but $n \neq 3$.

Thm. (Sarkovskii) Define an ordering on the natural numbers as follows:

$$3 < 5 < 7 < 9 < 11 \cdots$$
$$< 3\cdot 2 < 5\cdot 2 < 7\cdot 2 < \cdots$$
$$< 3\cdot 2^2 < 5\cdot 2^2 < 7\cdot 2^2 < \cdots$$
$$\vdots$$
$$< \quad 2^n < 2^{n-1} \cdots < 2^4 < 2^3 < 2^2 < 1$$

If an interval map has a point of period p then it has a point of period q where p < q with respect to the above ordering.