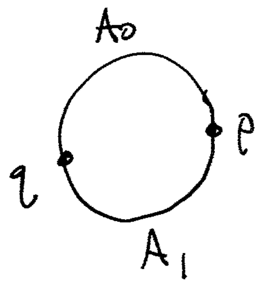
Thm. Let $f: S \circlearrowleft$ be an expanding map from the circle to itself with degree 2. Let Σ_2 be the sequence space on the symbols 0 and 1. Let $\sigma: \Sigma_2 \circlearrowleft$ be the left shift. Then there is a semi-conjugacy $h: \Sigma_2 \to S$ so that $h\sigma = fh$.

Proof. f has a unique fixed point. Denote it by p. $f^{-1}(p)$ contains 2 points one of which is p. Let q be the second point. Let $A_0 = (p, q)$ and $A_1 = (q, p)$. Thus $S = A_0 \cup A_1 \cup \{p, q\}$.



Fix an $n \geq 1$. Given $x \in S$ we can consider its trajectory of length n: $x, f(x), f^2(x) \dots f^{n-1}(x)$.

We associate to x a sequence of n-symbols $\omega_0 \dots \omega_{n-1}$ where:

$$\omega_j = \begin{cases} 0 & f^j(x) \in A_0 \\ 1 \quad \text{if} & f^j(x) \in A_1 \\ p & f^j(x) = p \\ q & f^j(x) = q \end{cases}$$

S decomposes into subsets with different itineraries $\omega_0 \dots \omega_{n-1}$. We want to understand this decomposition.

Note that itineraries follow the transitions given by:

$$\begin{tikzcd} 0 \arrow[loop above] \arrow[d, bend right] \arrow[dr] & & \\ & q \arrow[r] & p \arrow[loop right] \\ 1 \arrow[loop below] \arrow[u, bend right] \arrow[ur] & & \end{tikzcd}$$

A <u>vertex of level n</u> corresponds to a word $\omega_0 \dots \omega_{n-1}$ which contains a p or q.

For a word $\omega_0 \ldots \omega_{n-1}$ containing only 0's and 1's we write $A_{\omega_0 \ldots \omega_{n-1}}$ for the set of x with n-step itinerary $\omega_0 \ldots \omega_{n-1}$.

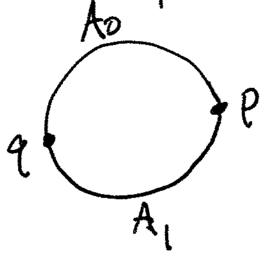
Claim. ① For any $\omega_0 \ldots \omega_{n-1} \in \Sigma_2$ $A_{\omega_0 \ldots \omega_{n-1}}$ is a non-empty open interval.

② The closure $\bar{A}_{\omega_0 \ldots \omega_{n-1}}$ is a closed interval with boundary consisting of vertices of level n.

③ $f^n | A_{\omega_0 \ldots \omega_{n-1}}$ is a homeomorphism onto $S - p$. $f^{n-1} | \bar{A}_{\omega_0 \ldots \omega_{n-1}}$ is a homeomorphism onto $\bar{A}_{\omega_{n-1}}$.

We prove the claim by induction.

For $n=1$ this is a statement about the decomposition and is easily seen to hold.



Assume the statement for n and we will prove it for $n+1$.

Let $w_0 \ldots w_n$ be a word of 0's and 1's of length $n+1$.

The definition of itinerary shows that $A_{w_0 \ldots w_n}$ is a sb subset of $A_{w_0 \ldots w_{n-1}}$.

It also shows that $A_{w_0 \ldots w_n}$ consists of those points x in $A_{w_0 \ldots w_{n-1}}$ for which $f^n(x) \in A_{w_n}$. By induction $f^n | A_{w_0 \ldots w_{n-1}}$ is a homeomorphism to $S-p$. Let $g = f^n | A_{w_0 \ldots w_{n-1}}$. $g^{-1}(A_{w_n})$ is an interval since A_{w_n} is an interval

Length estimate for intervals $A_{w_0 \ldots w_{n-1}}$.

Recall ~~that~~ f' is continuous and $|f'(x)|$ ~~[illegible]~~ >1

Since the circle is compact this implies

$|f'(x)| \geq |f'(x_0)| = K > 0$. So we have a uniform estimate. Now $f^n : A_{w_0 \ldots w_{n-1}} \to [p, p+1]$ is 1-1

so
$$\int_{A_{w_0 \ldots w_{n-1}}} (f^n)'(x)\,dx = (p+1) - p = 1$$

But
$$1 = \int_A (f^n)'(x)\,dx = \int_A f'(x) \cdot f'(f(x)) \cdot \ldots f'(f^{n-1}(x))\,dx$$
$$\geq \int_A K^n\,dx \,.$$
$$= K^n \cdot |A|$$

which gives $|A_w| \leq \frac{1}{K^n}$.

Now we construct the semi-conjugacy from Σ_2 to $S = \mathbb{R}/\mathbb{Z}$.

Let $\omega = \omega_0 \omega_1 \ldots$ be an infinite sequence.

Consider $\bigcap_{n=0}^{\infty} \bar{A}_{\omega_0 \ldots \omega_n}$.

These intervals are nested and decreasing in length to zero, so the intersection is a single point. Call it $h(\omega)$.

h is continuous. Given $\varepsilon > 0$ choose an n so that $\frac{1}{K^n} \le \varepsilon$. Let $\delta = \frac{1}{2^n}$. If $d(\omega, \omega') \le \frac{1}{2^n} = \delta$ then $\omega_0 \ldots \omega_{n-1} = \omega_0' \ldots \omega_{n-1}'$ so $h(\omega)$ and $h(\omega')$ are in $A_{\omega_0 \ldots \omega_{n-1}}$ which has diameter $\le \frac{1}{K^n} = \varepsilon$.

h is a semi-conjugacy. $f \circ h = h \circ \sigma$.

$\boxed{\text{Holder continuity but no better.}}$

$$f(h(\omega)) = f \bigcap_{n=0}^{\infty} \bar{A}_{\omega_0 \ldots \omega_n} = \bigcap_{n=0}^{\infty} f(\bar{A}_{\omega_0 \ldots \omega_n})$$

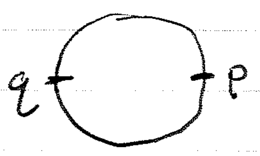
$$= \bigcap_{n=0}^{\infty} \bar{A}_{\omega_1 \ldots \omega_n}$$

$$= h(\sigma(\omega)).$$

~~When does h map two po~~ ω, ω'

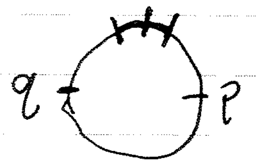
When do two points ~~map to~~ have the same image under h? $h(\omega)=h(\omega')$?

~~Either $\omega_0 = \omega_0'$~~



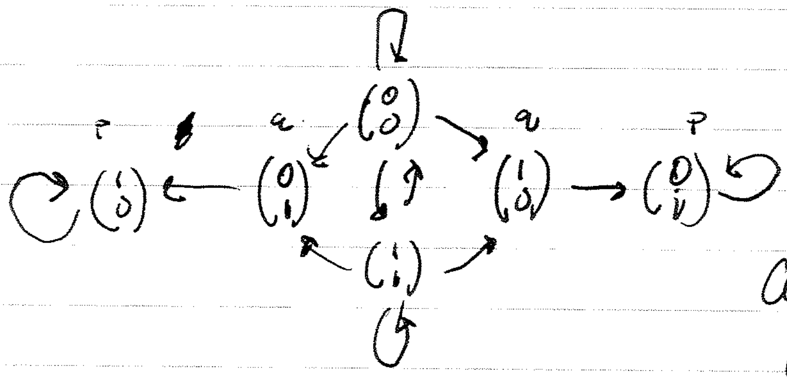
This means that for each finite sequence $\omega_0 \dots \omega_{n-1}$, $\omega_0' \dots \omega_{n-1}'$ the sets $A_{\omega_0 \dots \omega_{n-1}}$ and $A_{\omega_0' \dots \omega_{n-1}'}$ intersect.

There are 4 cases $\omega_0 = \omega_0' = 0$ or $\omega_0 = \omega_0' = 1$



or $\omega_0 \neq \omega_0'$ and $A_{\omega_0} \cap A_{\omega_0'} = q$
or $\omega_0 \neq \omega_0'$ and $\omega_0 \cap \omega_0' = p$.

~~If A_{ω_0}~~



Allowable transitions

(1) → (2) → (3)

Going from (2) to (3) the symbols switch.

We can say

Prop.

Two sequences $\omega = \omega_0\omega_1\ldots$ and $\omega' = \omega_0'\omega_1'\ldots$ have the same image under h if the sequence of pairs

$\binom{\omega_0}{\omega_0'}\ \binom{\omega_1}{\omega_1'}\ \cdots$ is compatible with the graph:

$$\begin{tikzcd}
 & & \binom{0}{0} \arrow[loop above] \arrow[dl] \arrow[dr] \arrow[d, bend right] & & \\
\binom{1}{0} \arrow[loop left] & \binom{0}{1} \arrow[l] & & \binom{1}{0} \arrow[r] & \binom{0}{1} \arrow[loop right] \\
 & & \binom{1}{1} \arrow[loop below] \arrow[ul] \arrow[ur] \arrow[u, bend right] & &
\end{tikzcd}$$

This graph also determines ~~the~~ when 2 binary expansions represent the same number.

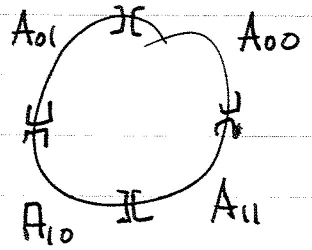
Example. $\begin{matrix} 0110000\ldots \\ 0101111\ldots \end{matrix}$

$\binom{0}{0}\ \binom{1}{1}\ \binom{1}{0}\ \binom{0}{1}\ \binom{0}{1}\ \ldots$

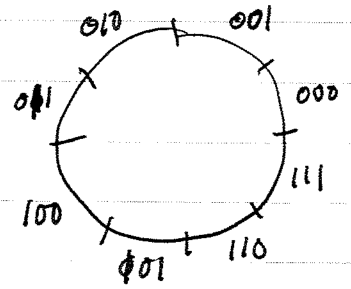
or $\begin{matrix} 0000\ldots \\ 1111\ldots \end{matrix}$

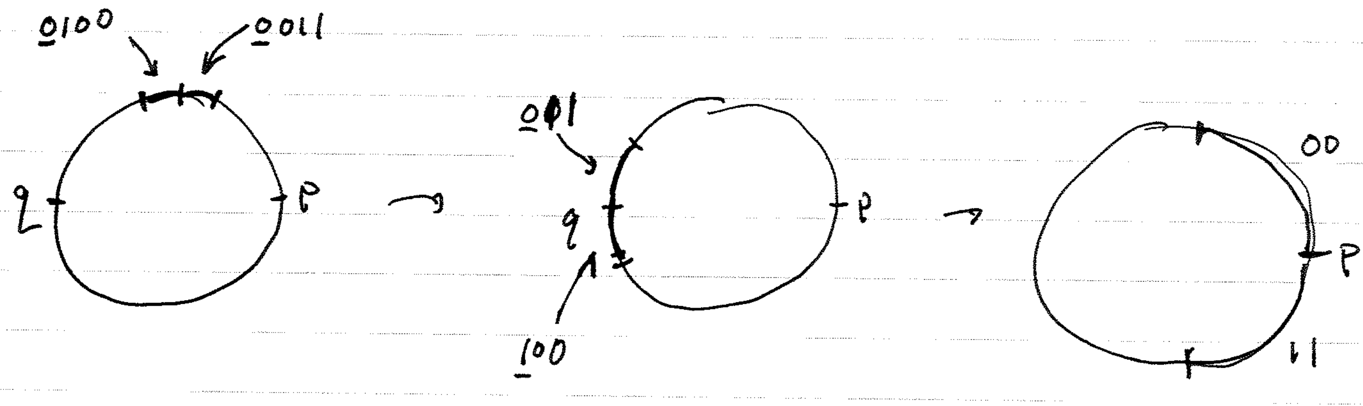
$\binom{0}{1}\ \binom{0}{1}\ \binom{0}{1}\ \binom{0}{1}\ \cdots$

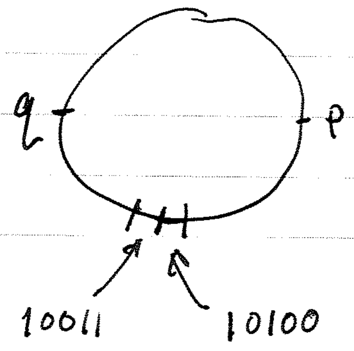
Recall that when we were coding intervals we described the ~~codes~~ itineraries of the interior points.



~~When~~ The coding of the boundary points is ambiguous. Instead of coding the boundary points we want to compare the codes of the two intervals on either side.







Prove that f_n and f_m are only top. conj. if $n=m$
$n \in \mathbb{Z} - \{0, \pm 1\}$.

Thm. If f and g are expanding maps of degree ± 2 then f and g are topologically conjugate.

$$\begin{tikzcd} & \Sigma_2 \arrow[dl] \arrow[dr] & \\ \mathbb{R}/\mathbb{Z} \arrow[rr, dashed] & & \mathbb{R}/\mathbb{Z} \end{tikzcd}$$

Quotient topology?

Very different from circle homeomorphisms

Thm If f is an expanding map then periodic points are dense, f is top. transitive.