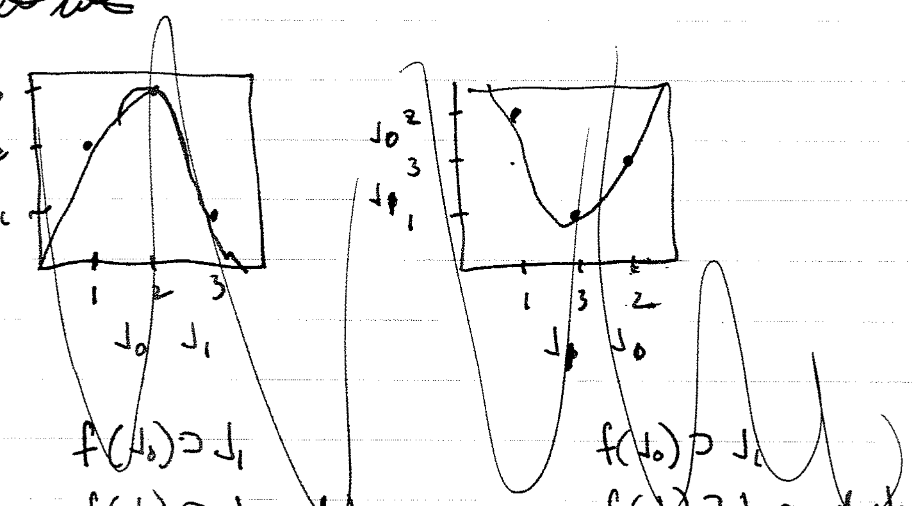
1975 1964

Thm. (Li-Yorke, Sarkovskii) Let f be a map from the interval to itself (not assumed to be unimodal). If f has a point of period 3 then f has periodic points of all periods.

(Period 3 $\Rightarrow$ chaos)

~~We will use~~



$f(J_0)\supset J_1$ $f(J_1)\supset J_0$ and J_1

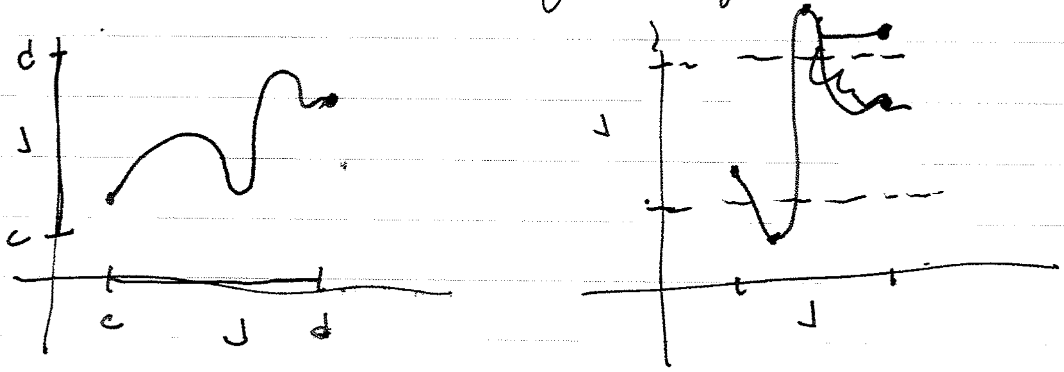
$f(J_0)\supset J_1$ $f(J_1)\supset J_0$ and J_1

We will need a series of lemmas which allows us to create and track periodic points

~~Lemma.~~ Any $f: I\to I$ is continuous.

Lemma. Let $J\subset I$ and assume that $f(J)\subset J$ or $f(J)\supset J$ then f has a fixed point in J.

Proof.



$f(J)\subset J$

Let $h(x)=f(x)-x$. ~~$h(c)$~~ $f(c)>c$, $f(d)<d$ so $h(c)\geq 0$ $h(d)\leq 0$. At some point in J $h(x)=0$. This is a fixed point.

Let $m = \min f|J$. Let $M = \max f|J$.

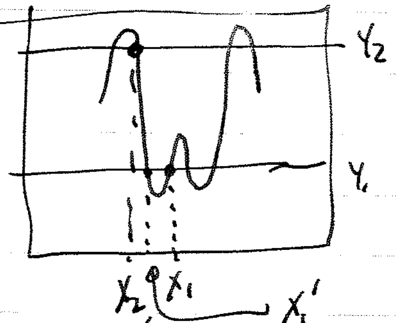
Pick x_0 with $f(x_0) = m$. Pick x_1 with $f(x_1) = M$.

Define $h(x) = f(x) - x$ on the interval $[x_0, x_1]$ (or $[x_1, x_0]$).

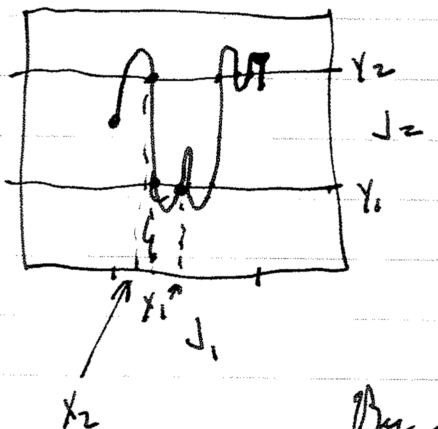
Since $f(x_0) = m \le x_0$ and $f(x_1) = M \ge x_1$ we have

$h(x_0) \le 0$ and $h(x_1) \ge 0$

Lemma. If $J_1, J_2 \subset I$ are closed intervals such that $J_2 \subset f(J_1)$ then there is a closed interval $J_0 \subset J_1$ such that $J_2 = f(J_0)$.



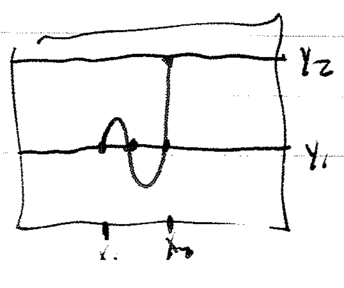
Proof. Say $J_2 = [y_1, y_2]$



Pick an $x_1 \in J_1$ that maps to y_1.

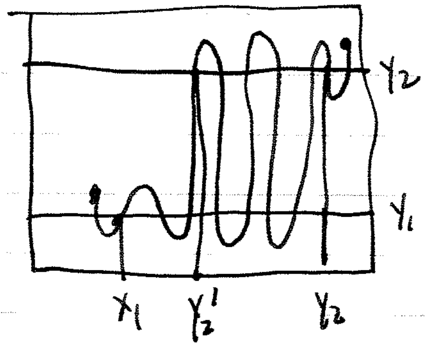
Among the points in $f^{-1}(y_2)$ find one closest to x_1 call it x_2.

By construction $f(x) \le y_2$ on $[x_1, x_2]$ (or $[x_2, x_1$



Now pick $x_1' \in [x_1, x_2]$ in $f^{-1}(y_1)$ and closest to x_2.

By construction $f(x) \ge y_1$ on $[x_1', x_2]$.



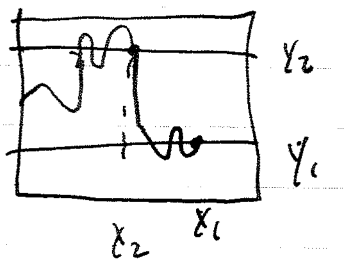
~~$f($~~ $f([a,b]) \supset [y_1, y_2]$

Let x_1 be a point mapping to y_1 and x_2 a point mapping to y_2.

Now replace x_2 by x_2' the point in $f^{-1}(y_2)$ nearest x_1.

We have $f \le y_2$ on $[x_1, x_2']$.

Replace x_1 be the point x_1' closest to x_2' in $f^{-1}(y_1)$.



Say $f(x_1) = y_1$. Let x_2 be the point in $f^{-1}(y_2)$ closest to x_1.

By construction on $[x_2, x_1]$, $f < y_2$.

Let x_1' be the closest point in $f^{-1}(y_1)$ in $[x_2, x_1]$.

On $[x_2, x_1']$ $f > y_1$, so $f([x_2, x_1']) \subset [y_1, y_2]$.

Cor. If $J \subset I$ is a closed interval with $J \subset f(J)$ then there is an ~~a~~ sequence of intervals $I_0 = J$, $I_{n+1} \subset I_n$ with $f(I_{n+1}) = I_n$ for $n \geq 0$.

Thus $y_1 \le f(x) \le y_2$ on $[x_1', x_2]$ and $f([x_1', x_2]) = [y_1, y_2]$.

Cor. Say we have intervals $J_0, J_1, J_2 \dots J_n$ with $f(J_i) \supset J_{i+1}$ then there is an $x \in J_0$ so that $f(x) \in J_1$, $f^2(x) \in J_2 \dots f^n(x) \in J_n$. In fact there is an $I_n \subset J_0$ so that $f^i(I_n) \subset J_i$ and $f^n(I_n) = J_n$.

Proof. ~~Dep~~ Apply the previous lemma repeatedly.

$f(J_0) \supset J_1$ so there is an $I_1 \subset J_0$ with $f(I_1) = J_1$.

$f^2(I_1) = f(J_1) \supset J_2$ so there is an $I_2 \subset I_1$ with $f^2(I_2) = J_2$.

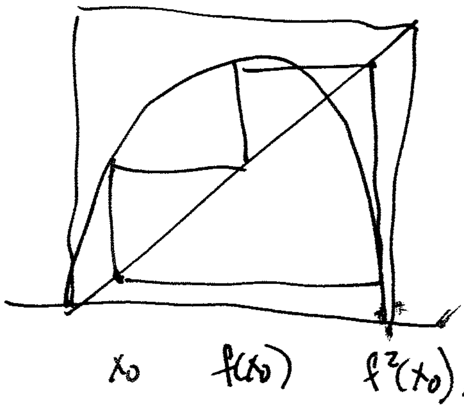
$f^3(I_2) = f(f^2(I_2)) = f(J_2) \supset J_3$ so there is an interval $I_3 \subset I_2$ with $f^3(I_2) = J_3$. etc.

(~~Note that it is useful here that we are not dealing~~ with ~~monotone unimodal~~ maps. ~~since iterates of uni~~

We now prove the theorem.

~~Let x be~~ Consider a point of period 3. Let x_0 be the smallest point on the orbit. $x_j := f^j(x_0)$.
Then either $x_0 < f(x_0) < f^2(x_0)$ or $x_0 < f^2(x_0) < f(x_0)$.
Consider the first case.

Let $J_1 = [x_0, f(x_0)]$, $J_2 = [f(x_0), f^2(x_0)]$.



$f(\partial J_1) = \{f(x_0), f^2(x_0)\}$ so $f(J_1) \supset J_2$.

$f(\partial J_2) = \{x_0, f^2(x_0)\}$ so $f(J_2) \supset J_1$ and J_2.

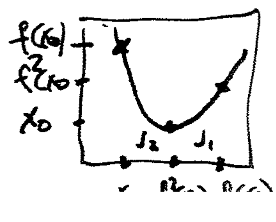
Diagram:

$$\begin{tikzcd} J_1 \arrow[r, bend left] & J_2 \arrow[l, bend left] \arrow[loop right] \end{tikzcd}$$

If $x_0 < f^2(x_0) < f(x_0)$ then let $J_2 = [x_0, f^2(x_0)]$

$J_1 = [f^2(x_0), f(x_0)]$.

In this case also we have



$$\begin{tikzcd} J_1 \arrow[r, bend left] & J_2 \arrow[l, bend left] \arrow[loop right] \end{tikzcd}$$

$$\begin{tikzcd} J_1 \arrow[r, bend left] & J_2 \arrow[l, bend left] \arrow[loop right] \end{tikzcd}$$

~~Exercise~~ To create a periodic point of period n ($n \neq 3$, $n \neq 1$) we choose a ~~sequence~~ cycle of intervals compatible with the graph above having length n:

$$\overset{0}{J_2} \to \overset{1}{J_2} \to \overset{2}{J_2} \cdots \to \overset{n-1}{J_1} \to \cancel{J_2}$$

$\underbrace{\qquad\qquad}_{n-1}$

~~2 2 1 2 2 1~~

$\omega = 2\ 2\ 2 \ldots 2\ 1$ ~~222~~

$\omega_0\ \omega_1 \ldots \omega_{n-2}$

By the lemma we can find an interval $I_n \subset J_2$ so that ~~$f^j(I_n) \subset \omega_j$ for $j = 0 \ldots n-1$ and $f^n(I_n) = J_2$~~

$$\begin{aligned} f^j(I_n) &\subset J_2 \text{ for } j = 0 \ldots n-2 \\ &\subset J_1 \text{ for } j = n-1 \\ &= J_2 \text{ for } j = n. \end{aligned}$$

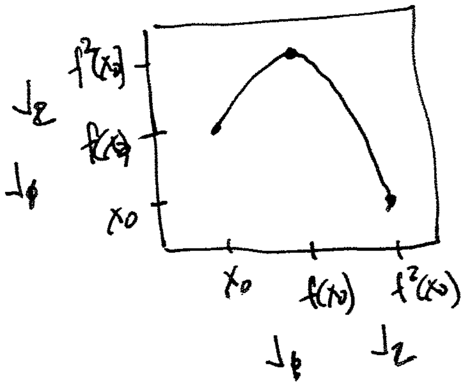
By previous lemma we have a ~~periodic~~ point $x \in I_n$ with $f^n(x) = x$.

Need to show that $f^m(x) \neq x$ for $m = 1 \ldots n-1$.

Say $f^m(x) = x$ then $f^{n-m}(x) \in J_1$ $= f^{n-m-1}(f^m(x)) = f^{n-1}(x)$

but $f^{n-m-1}(x) \in J_2$ by construction.

So $f^{n-m-1}(x) \in J_1 \cap J_2$ but this implies that x has period 3. This contradicts our assumption that $n \neq 3$



$$\begin{tikzcd} J_1 \arrow[r, bend left] & J_2 \arrow[l, bend left] \arrow[loop right] \end{tikzcd}$$

Either we constructed a point of period n or we constructed the orbit of x_0. All we showed was $f^n(x)=x$. What if $n=6$?

Coding for x_0 is ambiguous:

$$J_\phi \begin{matrix} \nearrow J_\phi \searrow \\ \searrow J_2 \nearrow \end{matrix} J_2 \to J_\phi \begin{matrix} \nearrow J_\phi \searrow \\ \searrow J_2 \nearrow \end{matrix} J_2 \to \dots$$

We cannot get $J_1 \to J_2 \to J_1 \to J_2 \dots$ or $\dots J_2\, J_2\, J_2 \dots$

Coding for the point we created:

$$\underbrace{\overset{x}{J_2}\quad \overset{f(x)}{J_2}\quad J_2 \dots J_2\quad \overset{f^{n-1}(x)}{J_1}}_{n}\quad \overset{f^n(x)}{J_2}\quad J_2 \dots J_2\quad J_1$$

The point we created is not ~~the point~~ on the orbit of x_0 if $n \neq 1$ or 3.