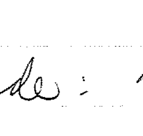
We are continuing our discussion of homeomorphisms of the circle viewed as dynamical systems.

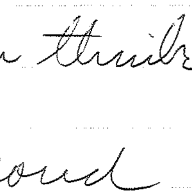
Aside: How do homeomorphisms of the circle arise? Say we are considering two periodic oscillators with different periods. We can think of phase space for the first oscillator as $X_1 = \mathbb{R}/\mathbb{Z}$ where $f_1^t(x) = x + v_1 t \mod 1$. We can think



of the phase space for the second oscillator as $X_2 = \mathbb{R}/\mathbb{Z}$ where $f_2^t(x) = x + v_2 t \mod 1$

Now the phase space for the system corresponding to the pair of oscillators is $X = X_1 \times X_2$

and the flow is $f^t(x_1, x_2) = (f_1^t(x_1), f_2^t(x_2))$.
$= (x_1 + v_1 t \mod 1,\ x_2 + v_2 t \mod 1)$

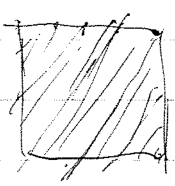


$f^t(x_1, x_2) = (x_1 + v_1 t,\ x_2 + v_2 t \mod 1)$

The period P of the first oscillator is $\frac{1}{v_1}$.

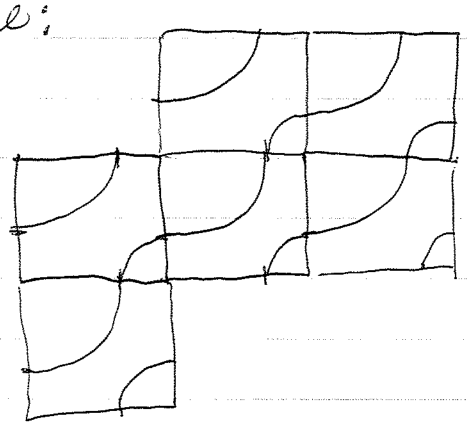
Say we flash a light every time the first oscillator comes back to its initial position. When we flash the light we observe the position of the second oscilator. At times the n-th flash the time is $np_1 = \frac{n}{v_1}$ and the position of the second oscelator is $f_2^t(0) \bmod 1 = f_2^{\frac{n}{v_1}}(0) = n\frac{v_2}{v_1} \bmod 1$. So it is determined by the map R_α where $\alpha = \frac{v_2}{v_1}$. If α is rational we only see finitely many possibilities for the position of the second oscilator. If α is irrational then the positions of the second oscilator are dense in the circle. In the first case we say that the two oscilators are "phase locked". Corresponds to the system having a closed orbit on the torus.

What happens if the oscilators are not independent? In this case R_α is replased by a circle diffeomorphism.

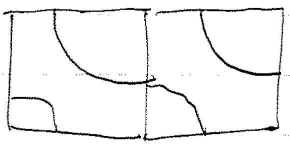


Last time we started our analysis of circle homeomorphisms by discussing lifts of homeomorphisms. F is a lift of f if F is continuous and: $F(x) \bmod 1$ depends only on $x \bmod 1$ and $F(x) \bmod 1 = f(x \bmod 1)$.

Picture:



This is a picture of an orientation preserving case. The orientation reversing case looks like:



We have:

(1) If $\tilde{F}$ and F are lifts of f then

$\tilde{F}(x) = F(x) + k$ for some k

(2) If f is orientation preserving then

$F(x+1) = F(x) + 1$

(3) If F is a lift of f then F^n is a lift of f^n for $n \in \mathbb{Z}$.

(4) If $F : \mathbb{R} \to \mathbb{R}$ is strictly monotone map such that $F(x+1) = F(x)+1$ for all $x \in \mathbb{R}$ then F is a lift of a circle homeomorphism.

Rotation numbers.

Let $f : \mathbb{R}/\mathbb{Z} \circlearrowleft$ be an orientation preserving homeomorphism and F a lift of f.

Define $$\rho(F) = \lim_{n\to\infty} \frac{F^n(x)}{n}$$

and $$\rho(f) = \rho(F) \bmod 1$$

Then $\rho(f)$ is the rotation number of f.

We need to show that $\rho(F)$ exists and is independent of x. We need to show that $\rho(F) \bmod 1$ depends only on f and not on the choice of a lift.

Example: $\rho(R_\alpha) = \alpha$.

Proof. $F(x) = x + \alpha$ is a lift.

$$\lim_{n\to\infty} \frac{F^n(x)}{n} = \lim_{n\to\infty} \frac{x+n\alpha}{n} = \alpha.$$

We have seen that the properties of α were the key to analyzing the dynamics of R_α. We will investigate the extent to which the properties of $f: \mathbb{R}/\mathbb{Z} \circlearrowleft$ are determined by $\rho(f)$ (after we show that $\rho(f)$ makes sense).

Assume f is orientation preserving. Let F be a lift of f.

Lemma. If $F(0) \geq 0$ then $F^k(0) \geq 0$ for all $k \geq 0$ in fact $F^k(0) \geq F^{k-1}(0) \geq \ldots F(0) \geq 0$.

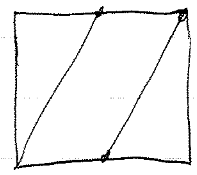
Proof by induction on k. Say $F^k(0) \geq F^{k-1}(0) \geq \ldots F(0) \geq 0$

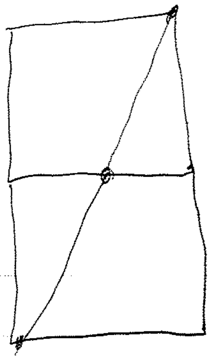
then since F is monotone increasing we have

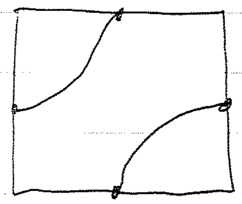
$F^{k+1}(0) \geq F^k(0) \geq \ldots F(0)$. Combine with the assumption

$F(0) \geq 0$.

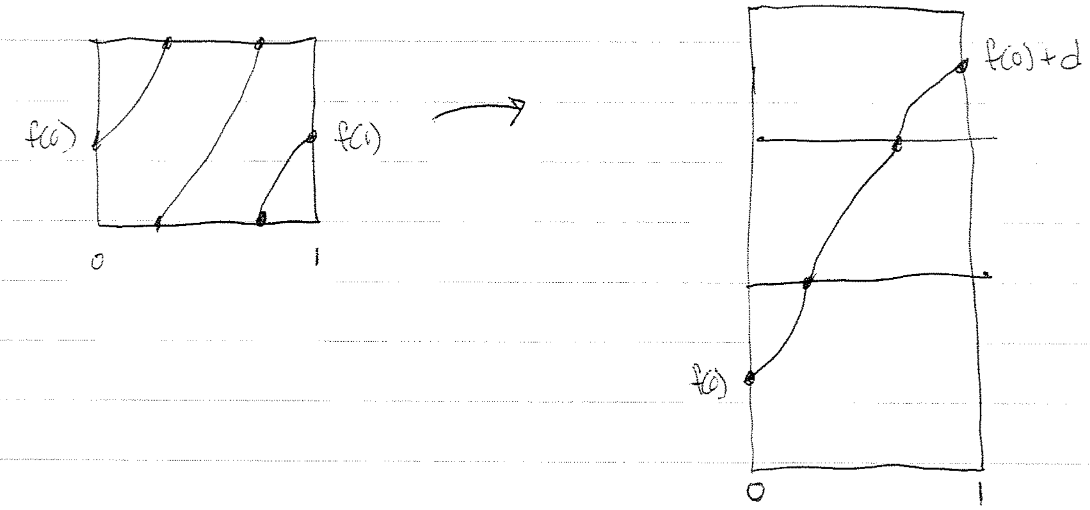
Similarly if $F(0) \leq 0$ then $F^k(0) \leq F^{k-1}(0) \leq \ldots \leq 0$.







<u>Any</u> continuous map $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ has a lift to a continuous map from $(0,1) \longrightarrow \mathbb{R}$.



This lift extends to a map $\bar{f}$ from $[0,1]$ to $\mathbb{R}$. The points $\bar{f}(0)$ and $\bar{f}(1)$ are equal mod 1 (equal to $f(0)$ mod 1) so $\bar{f}(1) - \bar{f}(0) \in \mathbb{Z}$. Call this number d. d is the topological degree of the map. $F(x+1) = F(x) + d$.

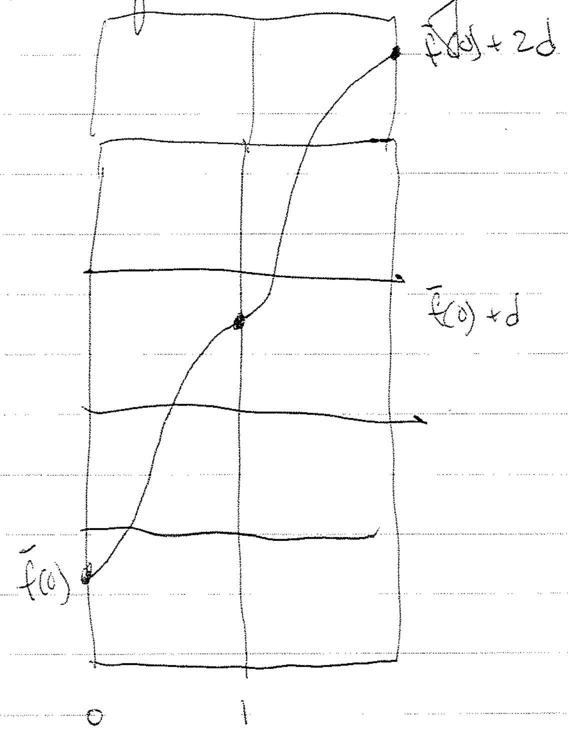
for $x \in [0,1]$,

If we define, $F(x+1) = \bar{f}(x) + d$ then the graphs of the two functions fit together to give a continuous function on $[0,2]$.

In general define $F(x+n) = \bar{f}(x) + dn$.

F will be a lift of f and will satisfy $F(x+n) = F(x) + dn$.

There is a relation between d and the number of inverse images of a point for f. In particular if each point has one pre-image then $d = \pm 1$. If f is orientation preserving then $d = +1$.



In this case

$$F(x+1) = F(x) + 1.$$

Def. Let $T(x) = x+1$.

$F(x+1) = F(x)+1$

We have $F \circ T = T \circ F$. It follows that $T^n F = F T^n$

so $F(x)+n = F(x+n)$.

Follows that

$(T^n F^m)^p = T^{np} F^{mp}$.

"$T^n F^m T^n F^m \ldots T^n F^m$"

Lemma. $\lim_{n \to \infty} \frac{F^n(0)}{n}$ exists.

Proof. Say that $F^n(0) \in [k, k+1]$ for $k \in \mathbb{Z}$.

Then $F^n(0) \geq k$ so $T^{-k} F^n(0) \geq 0$.

$T^{-k} F^n$ is a lift of f^n since F^n is a lift of f^n

$(T^{-k} F^n)^m (0) \geq 0$ for all m.

$T^{-km} F^{nm} (0) > 0$

$F^{nm}(0) \geq T^{km}(0) = km$.

Similarly $F^n(0) \leq k+1$ so $F^{nm}(0) \leq (k+1)m$.

$km \leq F^{nm}(0) \leq (k+1)m$

Thus

$$\frac{km}{mn} \leq \frac{F^{nm}(0)}{nm} \leq \frac{(k+1)m}{nm}$$

$$\frac{F^{nm}(0)}{nm} \in \left[\frac{k}{n}, \frac{k+1}{n}\right]$$

and $\left| \frac{F^n(0)}{n} - \frac{F^{nm}(0)}{nm} \right| \le \frac{1}{n}$.

So for any $n, m \in \mathbb{N}$ we get

$$\left| \frac{F^n(0)}{n} - \frac{F^m(0)}{m} \right| \le \left| \frac{F^n(0)}{n} - \frac{F^{nm}(0)}{nm} \right| + \left| \frac{F^m(0)}{m} - \frac{F^{nm}(0)}{nm} \right| \le \frac{1}{n} + \frac{1}{m}.$$

This shows that $\frac{F^n(0)}{n}$ is a Cauchy sequence:

Take $\varepsilon > 0$. For $n, m > \frac{2}{\varepsilon}$ we get

$$\left| \frac{F^n(0)}{n} - \frac{F^m(0)}{m} \right| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\frac{F^n(0)}{n}$ converges.

Scholium: If $F^n(0) \ge k$ then $\rho(F) \ge \frac{k}{n}$.

If $F^n(0) \le k$ then $\rho(F) \le \frac{k}{n}$.

$$\frac{F^{nm}(0)}{nm} \ge \frac{k}{n}$$

$$\Big\downarrow {\scriptstyle m \to \infty}$$

$$\rho(F)$$

Lemma. $\lim_{n \to \infty} \frac{F^n(x)}{n}$ exists for all x and has the same value for all x.

Proof. For some $m \in \mathbb{N}$ $\quad -m \le x \le m$.

Thus $\quad F^n(-m) \le F^n(x) \le F^n(m)$ since F^n is monotone increasing.

$$F^n T^{-m}(0) \le F^n(x) \le F^n T^m(0)$$

$$T^{-m} F^n(0) \le F^n(x) \le T^m F^n(0)$$

$$\frac{F^n(0) - m}{n} \le \frac{F^n(x)}{n} \le \frac{F^n(0) + m}{n}$$

So $\lim_{n \to \infty} \frac{F^n(x)}{n} = \lim_{n \to \infty} \frac{F^n(0)}{n}$.

Lemma. Let $\tilde{F}$ and F be lifts of f then

$\rho(\tilde{F}) = \rho(F)$.

Proof. $\tilde{F}(x) = F(x) + k$ for some k.

$$\lim_{n\to\infty} \frac{\tilde{F}^n(x)}{n} = \lim_{n\to\infty} \frac{(T^k F)^n(x)}{n} = \lim_{n\to\infty} \frac{T^{kn} F^n(x)}{n}$$

$$= \lim_{n\to\infty} \frac{F^n(x) + kn}{n} = \rho(F) + k.$$

So $\rho(\tilde{F}) = \rho(F) \bmod 1$.

For rotations R_α there was a connection between the rationality of α and periodic points of R_α. A similar connection holds for homeomorphisms.

$\rho(f^n) = n\rho(f)$:

Let F be a lift of f. Then F^n is a lift of f^n.

$$\rho(f^n) = \lim_{m\to\infty} \frac{F^{nm}(x)}{m} = n \lim_{m\to\infty} \frac{F^{nm}(x)}{nm} = n\cdot\rho(f).$$

uses convergence of $\frac{F^j(x)}{j}$

~~If f is orientation reversing then f^2 is or. preserving. Define $\rho(f) = \frac{1}{2}\rho(f^2)$.~~

Proposition. Let f be a homeomorphism of the circle then $\rho(f) = 0$ if and only if f has a fixed point.

Proof. Assume f is orientation preserving.

Assume $p = f(p)$. Let F be a lift of f. Since $f(p) = p$ we have $F(p) = p + n$ for some n.

Let $\tilde{F}(x) = F(x) - n$. Thus $\tilde{F}(p) = p + n - n = p$.

Now $\lim_{n\to\infty} \frac{\tilde{F}^n(p)}{n} = \lim_{n\to\infty} \frac{p}{n} = 0$.

Assume $\rho(f)=0$. Let F be a lift of f. Then $\lim_{n\to\infty} \frac{F^n(0)}{n} = m$ for some integer n. Let $\tilde{F}(x) = F(x) - n$. Then $\lim \frac{\tilde{F}^n(0)}{n} = 0$.

Claim that the sequence $\tilde{F}^n(0)$ is monotone.

If $\tilde{F}(0) > 0$ then $\tilde{F}(\tilde{F}(0)) > \tilde{F}(0)$ and $\tilde{F}^{n+1}(0) > \tilde{F}^n(0)$.

If $\tilde{F}(0) < 0$ then $\tilde{F}^{n+1}(0) < \tilde{F}(0)$.

Suppose $\tilde{F}^n(0)$ is ~~bounded~~ unbounded and increasing. There is an n_0 so that $\{\tilde{F}^{n_0}(0)\} > 1$. Arguing as before

$\tilde{F}^{n_0 m}(0) > m$ so $\rho(\tilde{F}) = \lim_{m\to\infty} \frac{\tilde{F}^{n_0 m}}{n_0 m} \geq \frac{m}{n_0 m} \geq \frac{1}{n_0}$.

This contradicts our assumption that $\rho(\tilde{F}) = 0$.

Conclude that $\tilde{F}^n(0)$ is bounded. Let $x_* = \lim_{n\to\infty} \tilde{F}^n(0)$.

$$F(x_*) = \lim_{n\to\infty} \tilde{F}^{n+1}(0) = \lim_{n\to\infty} \tilde{F}^n(0) = x_*.$$

So x_* is a fixed point for F and

$\pi x_* = x_* \bmod 1$

$f \pi x_* = \pi F x_* = \pi x_*$ is a fixed point for f.

Remark:
$\tilde{F}^k(x) > x + m \Rightarrow$
$\rho \tilde{F} > \frac{m}{k}$.