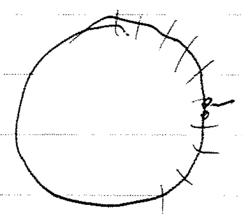
# Lecture 2.

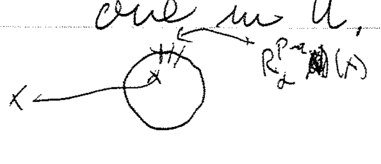
Proposition. If $\alpha$ is irrational then every orbit is dense in $\mathbb{R}/\mathbb{Z}$.

Proof. Assume $\alpha$ irrational. Given $x \in \mathbb{R}/\mathbb{Z}$ and an open set $U \subset \mathbb{R}/\mathbb{Z}$ we need to find an $n$ so that $R_\alpha^n(x) \in U$. Fix an $\varepsilon > 0$ so that $U$ contains an interval of length $\varepsilon$. Let $N > 1/\varepsilon$. Since $\alpha$ is irrational the points $S = \{x, R_\alpha(x) \ldots R_\alpha^N(x)\}$
(If $R_\alpha^i(x) = R_\alpha^j(x)$ ~~then~~ with $i<j$ then $R_\alpha^{j-i}(x) = x$ so $x$ is periodic).
are distinct. Divide the circle into $N$ intervals



of length $\frac{1}{N}$. By the pigeon hole principle one of these intervals contains two points in $S$, say $R_\alpha^p(x)$ and $R_\alpha^q(x)$.

Now $R_\alpha$ is an isometry so $d(R_\alpha^{-q}R_\alpha^q(x), R_\alpha^{-q}R_\alpha^p(x)) < \varepsilon$ so $d(x, R_\alpha^{p-q}(x)) < \varepsilon$. Now the iterates of $R_\alpha^{p-q}(x)$, $R_\alpha^{m(p-q)}(x)$ are $\varepsilon$ dense, there is at least one in any interval of length $\varepsilon$ so ~~take~~ there is one in $U$. Take $n = m(p-q)$ for appropriate $m$.



The initial digit is 7 if

$$7 \le m < 8 \quad \text{or}$$

$$\log_{10} 7 \le \log_{10} m < \log_{10} 8.$$

Let $I_7$ be this interval in $\mathbb{R}/\mathbb{Z}$.

Reformulated question:

Is $\log_{10} 2^n = n \log_{10} 2 \bmod 1$ in $I_7$?

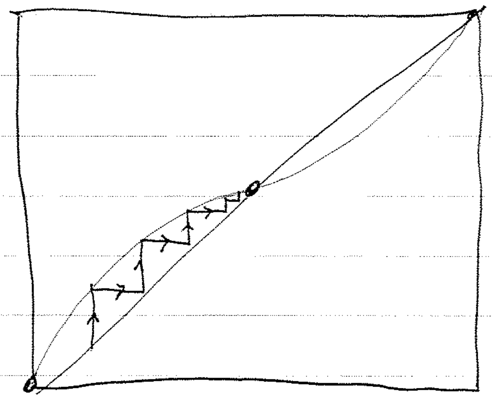
This is equivalent to $R^n_{\log_{10} 2} \in I_7$.

Since $\log_{10} 2$ is irrational this will be the case.

The same argument tells us that we can realize any finite sequence of initial digits.

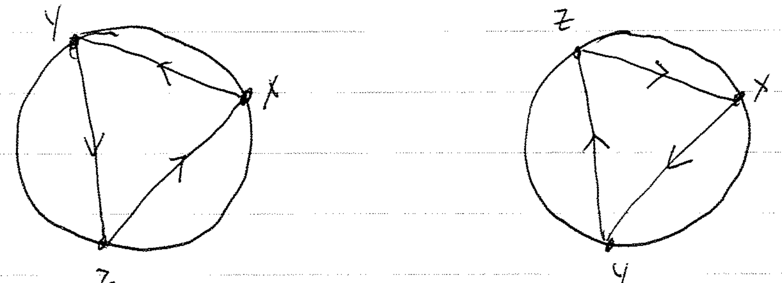
Our next topic is the dynamics of homeomorphisms of the circle. Unlike rotations homeomorphisms need not preserve distance between pairs of points. This leads to some new phenomenon.

Example. Let $f_c(x) = x + c + \frac{1}{10}\sin(2\pi x) \bmod 1$. For $c=0$ we can have isolated fixed points.



Here $0$ and $1/2$ are the only fixed points. Note that this behavior persists when $c$ is small but non-zero. In particular $c \mapsto f_c$ and $\alpha \mapsto R_\alpha$ have different behaviors.

Let $x, y, z$ be an ordered triple of points in the circle $\mathbb{R}/\mathbb{Z}$. Think of $\mathbb{R}/\mathbb{Z}$ as sitting inside $\mathbb{C}$. We say that $x, y, z$ determine a clockwise (anti-clockwise) orientation if the ~~triangular~~ boundary of the corresponding triangle in $\mathbb{C}$ has a clockwise (anti-clockwise) orientation.




anti-clockwise clockwise

Def. A homeomorphism $f : \mathbb{R}/\mathbb{Z} \circlearrowleft$ is orientation preserving (reversing) if $x, y, z$ and $f(x), f(y), f(z)$ have the same (opposite) orientations.

Notes on my personal website. George Kennison – Examples sheet

~~Let~~ Definition. Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a homeomorphism of the circle. A lift of $f$ is a map $F : \mathbb{R} \to \mathbb{R}$ such that $\pi \circ F = f \circ \pi$. Where $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the quotient map.

We can draw a commutative diagram

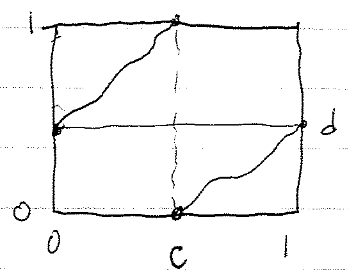
$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{R}/\mathbb{Z} & \xrightarrow{f} & \mathbb{R}/\mathbb{Z} \end{array}.$$

Proposition. Any homeomorphism $f$ has a lift $F$.

assume $f$ is orientation preserving.

Proof. Consider the ~~retraction~~ restriction of $f$ ~~to (0,1)~~ as a map from $(0,1)$ to $(0,1)$. Let $c = f^{-1}(0)$.

$(0,1) - \{c\}$ to $(0,1)$.



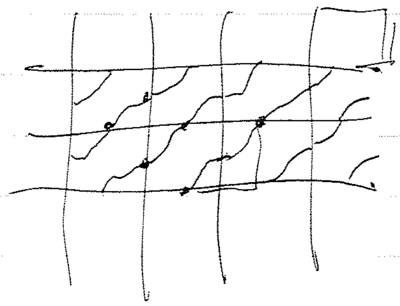
$f$ is monotone on the intervals $(0,c)$ and on $(c,1)$.

$f(0) = f(1) = d$.

We can extend the graph to ~~(c,1)~~ be monotone the points $\binom{c}{0}$ and $\binom{c}{1}$.

$\lim_{x\uparrow c} f(x) = 1$ and $\lim_{x\downarrow c} f(x) = 0$.

~~Start~~ Now think of this as a square tile and replicate across the plane. We get a family



of connected components. Each component is ~~a lift of F.~~ the graph of a lift of F.

A lift of F ~~has~~ the following properties:

(1) F is unique up to adding an integer

(2) $F(x+1) = F(x) + 1$

(3) If F is a lift of f then $F^n$ is a lift of $f^n$ for any $n \in \mathbb{Z}$.

Example: $F(x) = x + \alpha$ is a lift of $R_\alpha$.