Reading Group talk – Ch 4.

3 parts:

1) Coxeter groups & the Hecke algebra $\mathbb{H}_q$ (mostly recap)

2) Kazhdan - Lusztig basis / polynomials (with examples)

3) A very brief discussion of the Positivity of KL-polynomials via Soergel bimodules.

Let S be a finite set, let $\Gamma = (m_{ij})_{i,j \in S} \in (\mathbb{Z} \cup \{\infty\})^{n \times n}$, called the Coxeter matrix.

Coxeter group is $W := W(\Gamma) = \langle S \mid (xy)^{m_{xy}} = 1 \ \forall x, y \in S \rangle$.

<u>Example</u>

Constructing $A_2 = S_3 = \{e, (1\,2), (2\,3), (1\,3), (1\,2\,3), (1\,3\,2)\}$ as a Coxeter group.

$S = \{(1\,2), (2\,3)\}$, $\Gamma = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

$$W(\Gamma) = \langle \{(12), (23)\} \mid (12)^2 = 1,\ (23)^2 = 1,\ ((12)(23))^3 = 1,\ ((23)(12))^3 = 1 \rangle$$

$$= \{1, (12), (23), (12)(23), (23)(12), (12)(23)(12)\}$$

$$S_3 = \{e, (12), (23), (123), (132), (13)\}$$

So $S_3 \cong W(\Gamma)$.

The length function on (W, S)

Since S generates W, every $w \in W$ can be expressed in the form $w = s_1 s_2 \dots s_r$, where $s_i \in S$, not necessarily distinct,

Let $\ell(w) = len(w) = \min\{r \mid w = s_1 \dots s_r,\ s_i \in S\}$

Definition: If $w = s_1 \dots s_r$ with r as small as possible, '$s_1 s_2 \dots s_r$' is called a reduced expression.

Note: Reduced expressions are not unique, ~~but $\ell(w)$ is well defined~~.

Proposition: The function $\varepsilon : W \longrightarrow \{1, -1\}$, $w \longmapsto (-1)^{\ell(w)}$, is a group homomorphism, called the sign homom.

Example

$$(W, S) = (S_3, \{(12), (23)\}).$$

$\ell(e) = 0$, $\ell\big((12)(23)(12)(23)\big) = 2$

$\ell((12)) = \ell((23)) = 1$ since $(12)(23)(12)(23) = (23)(12)$.

$\ell((123)) = \ell((132)) = 2$

$\ell(13) = 3$

Bruhat ordering on (W, S).

- I kind of cheated here, in fact this definition is a theorem, since you have to prove it doesn't depend on which reduced expression you pick, but it is equivalent to any "proper definition" of Bruhat ordering you will find.

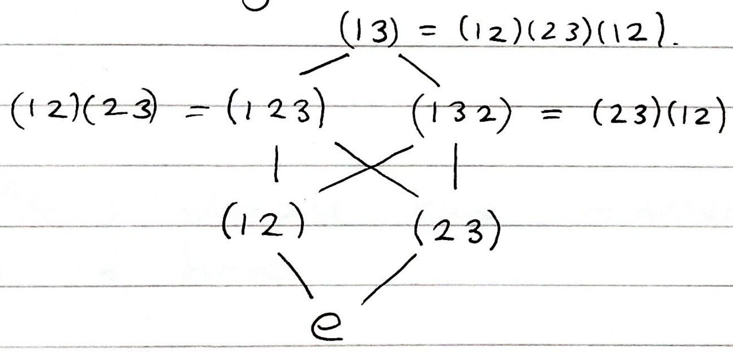
Definition: (Bruhat order) Let $w \in W$, fix a reduced expression $w = s_1 \dots s_r$.
We say $v \leq w$ iff $\exists$ ~~$\{i_1, \dots, i_k\} \subseteq \{1, \dots, r\}$~~ $\underline{e} = (e_1, \dots, e_r)$, $e_i \in \{0, 1\}$ such that
$v = s_1^{e_1} \dots s_r^{e_r}$

Example

$$(W, S) = (S_3, \{(1\,2), (2\,3)\})$$

$e = e$	$(1\,2\,3) = (1\,2)(2\,3)$
$(1\,2) = (1\,2)$	$(1\,3\,2) = (2\,3)(1\,2)$
$(2\,3) = (2\,3)$	$(1\,3) = (1\,2)(2\,3)(1\,2)$

On Hasse diagram:



The Hecke algebra ~~of (W, S)~~

$\Gamma = (m_{ij})_{i,j \in S}$ Coxeter matrix,
then $W = \langle S \mid (xy)^{m_{xy}} = 1 \ \forall\, x, y \in S \rangle$.

Define the Braid Monoid:

$$B(\Gamma) = \langle S \mid \underbrace{xyx\dots}_{m_{xy}\text{ symbols}} = \underbrace{yxy\dots}_{m_{xy}\text{ symbols}} \text{ for } x \neq y \rangle$$ (i.e. throw $x^2 = 1$ out.)

then $$\mathbb{H}_q := \mathbb{H}_q(\Gamma) = \mathbb{Z}[q^{\pm 1}]B(\Gamma) \Big/ \langle (x-q)(x+1) \rangle_{x \in S}$$

For $w = s_1 \dots s_r \in W$ a reduced expression,
Define $T_w := s_1 \dots s_r \in \mathbb{H}_q$, ~~let~~ ($T_1$ ~~the~~ is identity.)

- T_w doesn't depend on the reduced expression
- $\{T_w\}_{w \in W}$ form an A-basis for $\mathbb{H}_q$.

Another way to think about $\mathbb{H}_q$:

Consider the free $\mathbb{Z}[q^{\pm 1}]$ module with basis $\{T_w\}_{w \in W}$ with algebra structure given by:

Relation 1: $T_s T_w = T_{sw}$ if $\ell(sw) > \ell(w)$
Relation 2: $T_s^2 = (q-1)T_s + qT_1$.

for $s \in S$, $w \in W$.

<u>Lemma</u>

The basis elements have inverses, indeed, for $s \in S$ we have:

$$T_s^{-1} = q^{-1} T_s - (1 - q^{-1}) T_1.$$

<u>proof</u>

$$\begin{aligned} T_s^{-1} \cdot T_s &= \left(q^{-1} T_s - (1-q^{-1}) T_1\right) \cdot T_s \\ &= q^{-1} T_s^2 - (1-q^{-1}) T_s \\ &= q^{-1}\left[(q-1) T_s + q T_1\right] - (1-q^{-1}) T_s \\ &= \cancel{T_s} - \cancel{q^{-1} T_s} + T_1 - \cancel{T_s} + \cancel{q^{-1} T_s} = T_1. \end{aligned}$$

Showing $T_s \cdot T_s^{-1} = T_1$ is exactly the same ▨

If $w = s_1 \dots s_r$ reduced expression, then
$T_w = T_{s_1} \dots T_{s_r}$
$\Rightarrow T_w^{-1} = T_{s_r}^{-1} \dots T_{s_1}^{-1}$

As $\ell(w)$ grows, finding an explicit formula for T_w^{-1} gets messy.

The so-called "R-polynomials" give us a way to compute T_w^{-1} (or rather $(T_{w^{-1}})^{-1}$):

<u>Proposition</u>

Let $\varepsilon_w := (-1)^{\ell(w)}$, $q_w := q^{\ell(w)}$, then $\forall w \in W$

$$(T_{w^{-1}})^{-1} = \varepsilon_w q_w^{-1} \sum_{x \leq w} \varepsilon_x R_{x,w}(q) T_x$$

where $R_{x,w}(q) \in \mathbb{Z}[q]$ is of degree $\ell(w) - \ell(x)$ & $R_{w,w}(q) = 1$.

<u>Remark</u>: We have $R_{x,w}(q) \neq 0 \Leftrightarrow x \leq w$ so the Bruhat ordering is equivalent/determined by, the inversion in $\mathbb{H}_q$.

Too long, remove red band of stuff

<u>Deodhar's Formula</u>: a way to compute R polynomials.

Fix rex $w = s_1 \dots s_r$, so every $x \leq w$ appears as a subexpression.

Reformulation of subexpression: $\sigma = (1, \sigma_1, \dots, \sigma_r) \in W^{r+1}$

s.t $\sigma_j = \sigma_{j-1}$ or $\sigma_{j-1} s_j$ for all $1 \leq j \leq r$.
"REJECT s_j" "ACCEPT s_j"

$s_1 \ldots \hat{s_{i_1}} \ldots \hat{s_{i_q}} \ldots s_r \leftrightsquigarrow \sigma$ s.t $\sigma_j = \sigma_{j-1}$ for $j \in \{i_1, \ldots, i_q\}$.

Note: The subexpression $x \leq w$ is just σ_r.

Let:

- $n(\sigma) = q =$ "# of rejections."
- $m(\sigma) := \left|\{j : \sigma_{j-1} > \sigma_j\}\right| =$ "# times we accept s_j & $\sigma_{j-1} s_j < \sigma_{j-1}$."
- Distinguished subexpressions: (of $x \leq w$)

$$D(x) = \{\sigma : x = \sigma_r, \ \underbrace{\sigma_j \leq \sigma_{j-1} s_j \ \forall\, 1 \leq j \leq r}\}$$

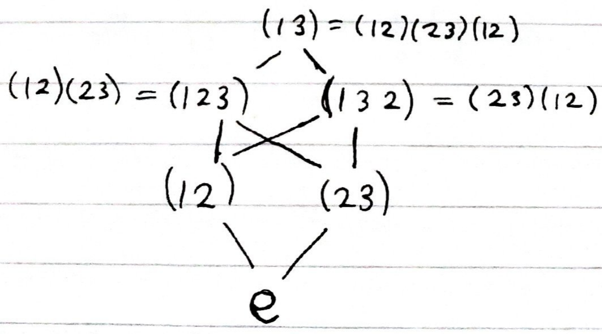
Require $\forall j$, each time we reject s_j, accepting instead would have given us a "greater word" w.r.t $\leq$.

Deodhar's formula:

$$R_{x,w} = \sum_{\sigma \in D(x)} (q-1)^{n(\sigma)} q^{m(\sigma)}.$$

<u>Example</u>

$$(W, S) = (S_3, \{(12), (23)\}),$$



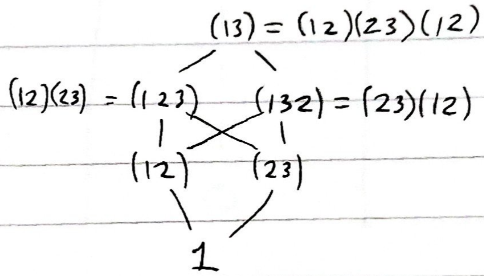
INCLUDE INSTEAD OF DEODHAR'S FORMULA

Formula for $R_{x,y}$: (recursive formula)

$$R_{x,y} = \begin{cases} 0 & , \text{ if } x \not\leq y \\ 1 & , \text{ if } x = y \\ R_{sx,sy} & , \text{ if } sx<x \ \& \ sy<y \\ R_{xs,ys} & , \text{ if } xs<x \text{ and } ys<y \\ (q-1)R_{sx,y} + qR_{sx,sy} & , \text{ if } sx>x \ \& \ sy<y. \end{cases}$$

Example

$(S_3, \{(12),(23)\})$



- $R_{(12),(12)(23)} \overset{\substack{\text{case 3} \\ s=(12)}}{=\!=} R_{1,(23)} \overset{\substack{\text{case 5} \\ s=(23)}}{=} (q-1)\underbrace{R_{(23),(23)}}_{=1} + q\underbrace{R_{(23),1}}_{=0}$
  $= q-1.$

Fact: If $x,w \in W$, $x \leq w$, $\ell(w)-\ell(x) = 1$ then $R_{x,w} = q-1.$

- $R_{(12),(12)(23)(12)} \overset{s=(12)}{=} R_{1,(23)(12)} = (q-1)R_{(23),(23)(12)} + qR_{(23),(12)}$
  $= (q-1)^2$

Fact: If $x,w \in W$, $x \leq w$, $\ell(w)-\ell(x) = 2$ then $R_{x,w} = (q-1)^2.$

- $R_{1,(12)(23)(12)} = (q-1)R_{(12),(12)(23)(12)} + qR_{(12),(23)(12)}$
  $= (q-1)^3 + q(q-1)$

Table: Subexpressions of $(13) = (12)(23)(12)$

	"REQUIRE NO REJECTIONS WHICH AVOIDED GOING DOWN"	"REJECTIONS"	"BAD ACCEPTS"
$\sigma = (1, \sigma_1, \sigma_2, \sigma_3)$	$D(-)$?	$n(\sigma)$	$m(\sigma)$
$\sigma = (1,1,1,1)$	$D(1)$	3	0
$\sigma = (1,(12),(12),(12))$	X (last rejection avoided $(12) \to 1$)	2	0
$\sigma = (1,1,(23),(23))$	$D((23))$	2	0
$\sigma = (1,1,1,(12))$	$D((12))$	2	0
$\sigma = (1,(12),(123),(123))$	$D((123))$	1	0
$\sigma = (1,(12),(12),1)$	$D(1)$	1	1
$\sigma = (1,1,(23),(132))$	$D((132))$	1	0
$\sigma = (1,(12),(123),(13))$	$D((13))$	0	0

$$R_{1,(13)} = (q-1)^3 + q(q-1)$$
$$R_{(12),(13)} = (q-1)^2$$
$$R_{(123),(13)} = q-1$$
$$R_{(13),(13)} = 1$$

Involution on $\mathbb{H}$ & K-L basis.

Define an involution (a ring automorphism of order 2):

$$i : \mathbb{H}_q \longrightarrow \mathbb{H}_q \quad \text{, extend linearly.}$$
$$q \longmapsto q^{-1}$$
$$T_w \longmapsto (T_{w^{-1}})^{-1}$$

Fact: This is a ring homomorphism, and in particular an involution.

$\boxed{\text{N.B:}}$ From now, we replace $\mathbb{Z}[q^{\pm 1}]$ with $\mathbb{Z}[q^{\pm \frac{1}{2}}]$.

Theorem

For each $w \in W$, there exists a unique element $C_w \in \mathbb{H}_q$ such that:

a) $i(C_w) = C_w$.

b) $C_w = \varepsilon_w q_w^{\frac{1}{2}} \sum_{x \leq w} \varepsilon_x q_x^{-1} P_{x,w}(q^{-1}) T_x$

where $P_{w,w} = 1$,
$P_{x,w}(q) \in \mathbb{Z}[q]$ w/ degree $\leq \frac{1}{2}(\ell(w) - \ell(x) - 1)$, $x < w$.

The $P_{x,y}$ for $x, y \in W$ are called Kazhdan-Lusztig polynomials.

Proof

- Uniqueness:

We show the uniqueness of: $\overline{P_{x,w}} = P_{x,w}(q^{-1})$

$$C_w = \sum_{x \leq w} a(x,w) \overline{P_{x,w}} T_x, \quad a(x,w) := \varepsilon_w \varepsilon_x q_w^{\frac{1}{2}} q_x^{-1}$$

assuming C_w satisfies a) & b).
This is equivalent to showing that $P_{x,w}$ can chosen in at most one way.

We fix a w, and proceed by induction on $\ell(w) - \ell(x)$, starting w/ $P_{w,w} = 1$.
Thus, we may assume $P_{y,w}$ are uniquely determined for $x < y \leq w$, we aim to show this forces choice of $P_{x,w}$.

$$C_w = \sum_{y \leq w} a(y,w) \overline{P_{y,w}} T_y$$

$$\overset{(a)}{=} i\left(\sum_{y \leq w} a(y,w) \overline{P_{y,w}} T_y\right) = \sum_{y \leq w} \varepsilon_w \varepsilon_y q_w^{-\frac{1}{2}} q_y^{+1} P_{y,w} (T_{y^{-1}})^{-1}$$

$$\overset{\text{Prop on } R \text{ polynomials } \& \, (T_{w^{-1}})^{-1}}{=} \sum_{x \leq y \leq w} \varepsilon_w q_w^{-\frac{1}{2}} \varepsilon_x R_{x,y} P_{y,w} T_x .$$

- Comparing T_x coefficients with this & original gives:

$$\varepsilon_w \varepsilon_x q_w^{\frac{1}{2}} q_x^{-1} \overline{P_{x,w}} = \varepsilon_w q_w^{-\frac{1}{2}} \sum_{x \leq y \leq w} \varepsilon_x R_{x,y} P_{y,w} .$$

GO STRAIGHT TO (+) SAY THAT WE CANCEL & MOVE $x=y$ TERM BUT DON'T WRITE IT.

- Cancelling signs & multiplying left & right by $q_x^{\frac{1}{2}}$ gives:

$$q_w^{\frac{1}{2}} q_x^{-\frac{1}{2}} \overline{P_{x,w}} = q_w^{-\frac{1}{2}} q_x^{\frac{1}{2}} \sum_{x \leq y \leq w} R_{x,y} P_{y,w} .$$

- Moving ~~$x \leq y$~~ $y = x$ term from RHS to LHS (using $R_{x,x} = 1$).

$$(+) \quad q_w^{\frac{1}{2}} q_x^{-\frac{1}{2}} \overline{P_{x,w}} - q_w^{-\frac{1}{2}} q_x^{\frac{1}{2}} P_{x,w} = q_w^{-\frac{1}{2}} q_x^{\frac{1}{2}} \sum_{x < y \leq w} R_{x,y} P_{y,w} .$$

We conclude by observing that since $\deg(P_{x,w}) \leq \frac{1}{2}(\ell(w) - \ell(x) - 1)$, $q_w^{\frac{1}{2}} q_x^{-\frac{1}{2}} \overline{P_{x,w}} \in \mathbb{Z}[q^{\frac{1}{2}}]$, $q_w^{-\frac{1}{2}} q_x^{\frac{1}{2}} P_{x,w} \in \mathbb{Z}[q^{-\frac{1}{2}}]$ and so RHS determines $P_{x,w}$.

- Existence:

Define '$\prec$' as follows:

- $x \prec w$ if $x < w$ & $P_{x,w}(q)$ has maximum allowed degree $\frac{1}{2}(\ell(w) - \ell(x) - 1)$.
- Let $\mu(x,w)$ be the coefficient of the highest power of q in $P_{x,w}$.

We proceed by induction on $\ell(w)$: we can compute C_w for $\ell(w) \leq 2$ easily by hand:

$C_1 := T_1$

$C_s,\ s\in S$: Using $T_s^{-1} = q^{-1}T_s - (1-q^{-1})T_1$
we see $i(T_s - qT_1) = q^{-1}T_s - T_1 = q^{-1}(T_s - qT_1)$
so set $C_s := q^{-\frac{1}{2}}(T_s - qT_1)$.

For C_{st}, we see that $C_{st} := C_sC_t$
$= q^{-1}(T_{st} - qT_s - qT_t + q^2T_1)$ satisfies a) & b).

Now, fix a $w\in W$. Find $s\in S$ s.t $\ell(sw) < \ell(w)$ and set $v = sw$, note that by assumption C_v has been constructed.

Obsere; $a(x,w) = \varepsilon_w\varepsilon_x q_w^{\frac{1}{2}} q_x^{-1}$

$$= -q^{\frac{1}{2}}\underbrace{\varepsilon_s\varepsilon_w}\varepsilon_x \underbrace{q^{-\frac{1}{2}}q_w^{\frac{1}{2}}} q_x^{-1}$$

to cancel out ε_s

$$= -q^{\frac{1}{2}}\varepsilon_{sw}\varepsilon_x q_{sw}^{\frac{1}{2}}q_x^{-1} = -q^{\frac{1}{2}}a(x,v)$$

Define $C_w := C_sC_v - \sum_{\substack{z \prec v \\ sz<z}} \mu(z,v)C_z$

- Since i ring homom, $i(C_w) = C_w$ ✓.

We wish to investigate the coefficient of T_x appearing in C_w for each fixed x, starting w/ $x = w$.

Recalling that $C_s := q^{-\frac{1}{2}}(T_s - qT_1)$, we see that T_w appears only in C_sC_v with coefficient :

$$q^{-\frac{1}{2}}a(v,v)\overline{P_{v,v}} = q^{-\frac{1}{2}}\cdot q_v^{\frac{1}{2}}\cdot q_v^{-1} = q_w^{-\frac{1}{2}},$$ which

since $T_sT_v = T_w$

agrees w/ expression for C_w in uniqueness part when $x = w$.

Next fix $x<w$, T_x can occur in 2 ways in $C_s C_v$ namely in $T_1 \cdot C_v$ (if $x \leq v$), or in $T_s \cdot C_v$ when T_s is multiplied by T_{sx} (if $sx \leq v$) Consider 2 cases:

<u>Case 1</u>: $x < sx$.

Thus $T_s T_{sx} = q T_x + (q-1) T_{sx}$, and so Coefficient of T_x in $q^{-\frac{1}{2}} T_s C_v$ is:

"factor s out"

$$q^{-\frac{1}{2}} \cdot q \cdot a(sx,v) \overline{P_{sx,v}} = q^{\frac{1}{2}} (-q^{-1}) a(x,v) \overline{P_{sx,v}}$$

use observation pg 10.

$$= q^{-1} a(x,w) \overline{P_{sx,v}}$$

OTOH, the coefficient of T_x in $-q^{\frac{1}{2}} T_1 C_v$ is:

$$-q^{\frac{1}{2}} a(x,v) \overline{P_{x,v}} = a(x,w) \overline{P_{x,v}}.$$

Combining this, the coefficient of T_x in $C_s C_v$ is:

$$q^{-1} a(x,w) \overline{P_{sx,v}} + a(x,w) \overline{P_{x,v}}. \qquad (x<sx).$$

<u>Case 2</u>: $sx < x$

DON'T INCLUDE THE WORKING

In this case $T_s T_{sx} = T_x$, $T_s T_x = q T_{sx} + (q-1) T_x$

So in $q^{-\frac{1}{2}} T_s C_v$ we have coefficients of T_x given by:

$$q^{-\frac{1}{2}} a(sx,v) \overline{P_{sx,v}} = q^{-\frac{1}{2}} (-q) a(x,v) \overline{P_{sx,v}} = a(x,w) \overline{P_{sx,v}}$$

$$\& \; q^{-\frac{1}{2}} (q-1) a(x,v) \overline{P_{x,v}} = (q^{-1} - 1) a(x,w) \overline{P_{x,v}}$$

OTOH, $-q^{\frac{1}{2}} T_1 C_v$ contains T_x with coefficient

$$-q^{\frac{1}{2}} a(x,v) \overline{P_{x,v}} = a(x,w) \overline{P_{x,v}}$$

Combining this, the coefficient of T_x in $C_s C_v$ is:

$$a(x,w) \overline{P_{sx,v}} + q^{-1} a(x,w) \overline{P_{x,v}}.$$

Lastly, the coefficient of T_x in $-\sum \mu(z,v)\, C_z$ is

$$-\sum \mu(z,v)\, a(x,z)\, \overline{P_{x,z}}$$

$$\|$$

$$-\sum \mu(z,v)\, q_z^{\frac{1}{2}} q_w^{-\frac{1}{2}}\, a(x,w)\, \overline{P_{x,z}}$$

since $v = sw$

$z < w \Rightarrow \varepsilon_z = -\varepsilon_v = \varepsilon_w$

$a(x,z) = \varepsilon_x \varepsilon_z q_z^{\frac{1}{2}} q_x^{-1}$

$= \varepsilon_x \varepsilon_w q_z^{\frac{1}{2}} q_w^{-\frac{1}{2}} q_w^{\frac{1}{2}} q_x^{-1}$

$= q_z^{\frac{1}{2}} q_w^{-\frac{1}{2}} a(x,w).$

Using this, we may now express C_w in the form $\sum_{x \le w} a(x,w)\, \overline{P_{x,w}}\, T_x$ where:

$$\overline{P_{x,w}} = \begin{cases} q^{-1}\overline{P_{sx,v}} + \overline{P_{x,v}}, & x < sx \\ \overline{P_{sx,v}} + q^{-1}\overline{P_{x,v}}, & sx < x \end{cases}$$

$$- \sum_{\substack{z < v \\ sz < z}} \mu(z,v)\, q_z^{\frac{1}{2}} q_w^{-\frac{1}{2}}\, \overline{P_{x,z}}$$

$$\Rightarrow P_{x,w} = q^{1-c} P_{sx,v} + q^{c} P_{x,v} - \sum_{\substack{z < v \\ sz < z}} \mu(z,v)\, q_z^{-\frac{1}{2}} q_w^{\frac{1}{2}} P_{x,z} \qquad (\#)$$

where $c = \begin{cases} 0 & \text{if } x < sx \\ 1 & \text{if } sx < x \end{cases}$, and $v = sw$.

Some thought shows that $P_{x,w}$ satisfies degree constraints, however the case when $c = 1$ $(sx < x)$ is worth considering explicitly:

Here, $q P_{x,v}$ could be of degree $1 + \frac{1}{2}(\ell(v) - \ell(x) - 1)$

$= \frac{1}{2}(\ell(w) - \ell(x))$ TOO BIG

However, then we have $x < v$ ($\&$ $sx < x$) so there is a term for $z = x$ in the "$\Sigma \ldots$", the term is precisely $-\mu(x,v)\, q^{\frac{1}{2}(\ell(w) - \ell(x))}$ so this cancels the highest term in $q P_{x,v}$, so the degree condition is met. $\boxtimes$

Example

Recall ~~Last~~:
$$R_{x,w} = q-1 \quad \text{if } \ell(w)-\ell(x)=1$$
$$R_{x,w} = (q-1)^2 \quad \text{if } \ell(w)-\ell(x)=2$$

Use (†): $$q_w^{\frac{1}{2}} q_x^{-\frac{1}{2}} \overline{P_{x,w}} - q_w^{-\frac{1}{2}} q_x^{\frac{1}{2}} P_{x,w} = q_w^{-\frac{1}{2}} q_x^{\frac{1}{2}} \sum_{x<y\leq w} R_{x,y} P_{y,w}$$

Claim: If $x<w$, $\ell(w)-\ell(x)=1$, $P_{x,w}=1$.

[[$\deg P_{x,w} \leq \frac{1}{2}(1-1)=0$, so let $P_{x,w}=\lambda$, $\lambda\in\mathbb{Z}$.

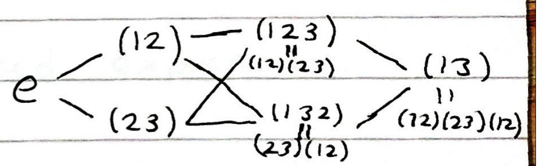
$$(\dagger) \Rightarrow \lambda q^{\frac{1}{2}} - \lambda q^{-\frac{1}{2}} = q^{-\frac{1}{2}} R_{x,w} P_{w,w} = q^{-\frac{1}{2}}(q-1) = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$$]]

Claim: If $x<w$, $\ell(w)-\ell(x)=2$, $P_{x,w}=1$

[[$\deg P_{x,w} \leq \frac{1}{2}(2-1)=\frac{1}{2} \Rightarrow P_{x,w}=\lambda$, $\lambda\in\mathbb{Z}$.

$$(\dagger) \Rightarrow \lambda q - \lambda q^{-1} = q^{-1}\Big(\overbrace{\sum_{x<y<w}(q-1)}^{\text{2 of these.}} + \overbrace{(q-1)^2}^{y=w \text{ term.}}\Big)$$
$$= q^{-1}(2q-2+q^2-2q+1) = q-q^{-1}.$$]]

If $(W,S) = (S_3, \{(12),(23)\})$



Need $P_{1,(13)}$:

$$q^{3/2}\overline{P} - q^{-3/2}P = q^{-3/2} \sum_{\substack{y\in S_3 \\ y\neq 1}} R_{1,y} P_{y,(13)}$$
$$= q^{-3/2}\Big(\underbrace{2(q-1)}_{\ell(y)=1} + \underbrace{2(q-1)^2}_{\ell(y)=2} + \underbrace{R_{1,(13)}}_{\ell(y)=3 \text{ ie } y=(13)}\Big)$$
$$= \dots = q^{3/2} - q^{-3/2}$$

$\Rightarrow P_{1,(13)} = 1.$

Conclusion: All K-L polynomials for S_3 are 1.

A brief word on Soergel-bimodules & positivity of K. L. polynomials.

The "v-version" of $\mathbb{H}$ is better here.

- Replace T_s with $H_s := q^{-\frac{1}{2}} T_s = v T_s \quad (v = q^{-\frac{1}{2}})$
- Relation 2: $H_s^2 = (v^{-1} - v) H_s + H_1$
- K-L basis: $\underline{H}_w = H_w + \sum_{x<w} h_{x,w} H_x, \quad h_{x,w} \in v\mathbb{Z}[v]$
- Recall from Adam's talk, for any (W, S) we have a "natural representation" V, $\dim V = |S|$.
- Let $R = \mathrm{Sym}^{\bullet}(V^*)$ with V^* in degree 2.
- For each $s \in S$, we define a graded R-bimodule $B_s = R \otimes_{R^s} R(1)$, where $R^s = \{ f \in R : s \cdot f = f \}$, grading shift (1) means $1 \otimes 1$ is in degree -1.
- Let $w = s_1 s_2 \dots s_r$ be reduced expression. Define the R-bimodule:

$$BS(w) = B_{s_1} \otimes_R B_{s_2} \otimes_R \dots \otimes_R B_{s_r}$$

Bott Samuelson bimodules.

- For any $w \in W$, $\exists !$ up to isomorphism bimodule B_x which occurs as a direct summand of $BS(w)$ but not in $BS(w')$ for $l(w') < l(w)$.

Defn: A Soergel bimodule is a direct summand of $(\oplus, \otimes, (+/-))$ of Bott-Samuelson bimodules. Denote category of Soergel bimodules as $\mathbb{S}\mathrm{Bim}$.

- The bimodules $\{B_x\}_{x \in W}$ give representatives of all indecomposable Soergel bimodules up to degree shifts. up to isomorphism

- One can define a character $ch : [\mathbb{S}Bim] \longrightarrow \mathbb{H}$ split Grothendieck group.

- It is a $\mathbb{Z}_{\geq}[v^{\pm 1}]$ linear combination of $\{H_x\}_{x \in W}$

Theorem (Soergel's conjecture)

For all $w \in W$, $ch(B_x) = \underline{H}_x$

Corollary

The Kazhdan-Lusztig polynomials $h_{x,y} \in \mathbb{N}_0[v]$ for all $x, y \in W$.