The relative Lie algebra cohomology of the Weil representation of $\mathfrak{o}(p,1) \times \mathfrak{sl}(2)$ with coefficients in $\mathrm{Sym}^l(V)$

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Declaration

This piece of work is a result of my own work except where it forms an assessment based on group project work. In the case of a group project, the work has been prepared in collaboration with other members of the group. Material from the work of others not involved in the project has been acknowledged and quotations and paraphrases suitably indicated.

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1 Introduction

1.1 Background and motivation

Lie algebra cohomology was introduced as a way to study properties of Lie groups. This is achieved by relating the so-called de Rham cohomology of the Lie group to the Lie algebra cohomology of the associated Lie algebra. A sketch of the relation is as follows (see e.g. [1, Section 2.1]).

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then one can define the co-chain complex

$$\Omega^\bullet(G)$$

consisting of differential forms on G, with the exterior derivative as the differential. This is known as the de Rham complex of G. Now, we have a diffeomorphism on G given by left translation,

$$L_g : G \to G, \;\; L_g(h) := gh$$

for all $g \in G$, $h \in G$. This induces a linear map on the tangent space of G at each point $h \in G$:

$$(dL_g)_h : T_h G \to T_{gh} G.$$

Now, given a differential $q-$form, $\omega \in \Omega^q(G)$, that is, for each $g \in G$, ω gives a map

$$\omega_g : \bigwedge^q \mathrm{T}_g G \to \mathbb{R}$$

we have a pullback on ω induced by dL_g given by

$$\left(dL_g^* \cdot \omega\right)_h (x_1, ..., x_q) := \omega_{gh}(dL_g(x_1), ..., dL_g(x_q)).$$

where $(x_1, ..., x_q) \in \bigwedge^q T_h G$. We say $\omega \in \Omega^\bullet(G)$ is left invariant if $dL_g^* \cdot \omega = \omega$ for all $g \in G$. We denote the space of left invariant differential $q-$forms by $\Omega^q_L(G)$. In fact one obtains a sub-complex

$$\Omega^\bullet_L(G) \subset \Omega^\bullet(G).$$

Let $\mathfrak{g} = \mathrm{Lie}(G)$, then the so-called Chevalley-Eilenburg complex of $\mathfrak{g}$ is

$$C^\bullet(\mathfrak{g}) = \mathrm{Hom}(\bigwedge^\bullet \mathfrak{g}, \mathbb{R}),$$

with differential given as in Section 2.1. Given $\omega \in C^q(\mathfrak{g})$, we define a differential $q-$form, which we denote $F\omega$, as follows;

$$(F\omega)_g(\underbrace{x_1, ..., x_q}_{\in \bigwedge^q T_g G}) := \omega(\underbrace{dL_{g-1}(x_1), ..., dL_{g-1}(x_q)}_{\in \bigwedge^q T_e G \cong \bigwedge^q \mathfrak{g}}).$$

One finds that $F\omega$ is a left invariant differential form. Moreover we have an isomorphism

$$C^\bullet(\mathfrak{g}) \cong \Omega^\bullet_L(G).$$

Now, let K be a closed connected subgroup of G, let $D = {}^{G}\!/_{K}$, $\mathfrak{k} = \mathrm{Lie}(K)$, and let (ρ, V) be a smooth representation of G. One can see in e.g. [2, Chapter I, Section 1.6] that the above isomorphism of complexes can be extended to an isomorphism between

$$C^{\bullet}(\mathfrak{g}, \mathfrak{k}; V) = \mathrm{Hom}_{\mathfrak{k}}\left(\bigwedge\nolimits^{\bullet} \mathfrak{g}/_{\mathfrak{k}}, V\right) = \left[V \otimes \bigwedge\nolimits^{\bullet} \mathfrak{g}/_{\mathfrak{k}}{}^{*}\right]^{\mathfrak{k}}$$

and

$$\Omega^{\bullet}\Big(D; V\Big)^{G} = \Omega^{\bullet}\Big({}^{G}\!/_{K}; V\Big)^{G},$$

the space of G-invariant $V-$valued differential forms on D, where the group G operates on $\omega \in \Omega^{q}(D, V)$ via

$$\big(g \cdot \omega\big)_{h}(x_1, ..., x_q) = \rho(g^{-1})\, \omega_{gh}\Big(dL_g(x_1), ..., dL_g(x_q)\Big)$$

for all $g \in G$, $h \in D$ and $(x_1, ..., x_q) \in \bigwedge^{q} T_h D$. Analogous to before, we say ω is $G-$invariant if $g \cdot \omega = \omega$ for all $g \in G$. Moreover, $\big($see [2, Chapter I, Section 5]$\big)$ if G is a Lie group with finitely many connected components; K is a maximal compact subgroup of G with identity component K_0; and V is a smooth representation of G, then we define

$$C^{\bullet}(\mathfrak{g}, K; V) := \mathrm{Hom}_{K}\left(\bigwedge\nolimits^{\bullet} \mathfrak{g}/_{\mathfrak{k}}, V\right) = \left[V \otimes \bigwedge\nolimits^{\bullet} \mathfrak{g}/_{\mathfrak{k}}{}^{*}\right]^{K}$$

where K acts via the adjoint action on $\mathfrak{g}/_{\mathfrak{k}}$. We have

$$C^{\bullet}(\mathfrak{g}, K; V) = C^{\bullet}(\mathfrak{g}, \mathfrak{k}; V)^{K/K_0} \quad \text{and} \quad H^{\bullet}(\mathfrak{g}, K; V) = H^{\bullet}(\mathfrak{g}, \mathfrak{k}; V)^{K/K_0}.$$

In this report we will consider the case where $G = SO_0(p, 1)$, the connected component of the identity of $O(p, 1)$, where $O(p, 1)$ is the indefinite orthogonal group of the real vector space V with signature $(p, 1)$; $K = SO(p) \times SO(1)$, the maximal compact (connected) subgroup of G; $\mathfrak{g} = \mathfrak{o}(p, 1)$; $\mathfrak{k} \cong \mathfrak{so}(p)$; and our representation space is $\mathcal{F} \otimes \mathrm{Sym}^{l}(V)$, where $\mathcal{F} = \mathbb{C}[z_1, ..., z_p, z_{p+1}]$ is the Fock model of the Weil representation (see Section 2.2). There exists an isomorphism between $\mathcal{F}$ and $\mathbf{S}(V) \subset \mathscr{S}(V)$, where $\mathscr{S}(V)$ denotes the space of "rapidly decaying" functions $f : V \to \mathbb{C}$ given by sending $1 \in \mathcal{F}$ to (a multiple of) the Gaussian $\varphi_0 := e^{-\pi \sum_{i=1}^{p+1} v_i^2}$ and z_α to $\big(v_\alpha - \frac{1}{2\pi}\frac{\partial}{\partial v_\alpha}\big)\varphi_0$. This gives an isomorphism

$$C^{q}\big(\mathfrak{g}, K; \mathcal{F}\big) \cong \left[\mathbf{S}(V) \otimes \bigwedge\nolimits^{q} \mathfrak{g}/_{\mathfrak{k}}{}^{*}\right]^{K} \cong \Omega\Big({}^{G}\!/_{K};\ \mathbf{S}(V)\Big)^{G}.$$

We may extend this to an isomorphism

$$C^{q}\Big(\mathfrak{g}, K; \mathcal{F} \otimes \mathrm{Sym}^{l}(V)\Big) \cong \Omega\Big({}^{G}\!/_{K};\ \mathbf{S}(V) \otimes \mathrm{Sym}^{l}(V)\Big)^{G}.$$

One may use this isomorphism, along with the so called "theta correspondence" to construct automorphic forms on suitable quotients of ${}^{SL(2,\mathbb{R})}\!/_{U(n)}$. For more details see [3, Section 1.0.9, The theta correspondence] and [4].

1.2 Statement of results and overview

Let $G = SO_0(p,1)$; $K = SO(p) \times SO(1)$; $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{o}(p,1)$; $\mathfrak{k} = \mathrm{Lie}(K) \cong \mathfrak{so}(p)$. Furthermore, let $\mathcal{F}$ be the Fock model of the Weil representation and V be the standard representation of G. In this report we will reproduce the following result found in [3, Theorem 4.1.1, k=1], as well as give a result on the $\mathfrak{sl}(2)-$module structure of the cohomology groups.

Theorem 1.1.

$$H^i(\mathfrak{g}, K; \mathcal{F}) = \begin{cases} \bigoplus_{k=0}^{\infty} \left\langle \left[(r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \omega_\alpha \right] \right\rangle & \text{if } i = 1 \\ \bigoplus_{l=0}^{\infty} \left\langle \left[z_{p+1}^l \otimes \Omega \right] \right\rangle \bigoplus \bigoplus_{k=1}^{\infty} \left\langle \left[(r^2)^k \otimes \Omega \right] \right\rangle & \text{if } i = p \\ 0 & \text{otherwise,} \end{cases}$$

where $\Omega := \omega_1 \wedge \omega_2 \wedge ... \wedge \omega_p$.

Theorem 1.2. *The cohomology groups have the following $\mathfrak{sl}(2)-$module structure (notation from Theorem 3.10):*

1. $\bigoplus_{k=0}^{\infty} \left\langle \left[(r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \omega_\alpha \right] \right\rangle \simeq (\,[\,\circ\,)$.

2. $\bigoplus_{l=0}^{\infty} \left\langle \left[z_{p+1}^{2l+1} \otimes \Omega \right] \right\rangle \simeq \begin{cases} (\,\circ\,]\,\circ\,]) & \text{if } p \text{ is odd.} \\ (\,\circ\,]) & \text{if } p \text{ is even.} \end{cases}$

3. $\bigoplus_{l=0}^{\infty} \left\langle \left[z_{p+1}^{2l} \otimes \Omega \right] \right\rangle \bigoplus \bigoplus_{k=1}^{\infty} \left\langle \left[(r^2)^k \otimes \Omega \right] \right\rangle \simeq \begin{cases} (\,\circ\,) & \text{if } p \text{ is odd.} \\ (\,\circ\,]\,\circ\,) & \text{if } p \text{ is even.} \end{cases}$

Furthermore, we will prove analogous results for $H^\bullet(\mathfrak{g}, K; \mathcal{F} \otimes V)$. These are as follows:

Theorem 1.3.

$$H^i\big(\mathfrak{g}, K; \mathcal{F} \otimes V\big) = \begin{cases} \bigoplus_{k=0}^{\infty} \left\langle \left[(r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha z_\beta \otimes \omega_\alpha \otimes e_\beta \right] \right\rangle & \text{if } i = 1 \\ \bigoplus_{l=0}^{\infty} \left\langle \left[z_{p+1}^l \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \right] \right\rangle \bigoplus \bigoplus_{k=1}^{\infty} \left\langle \left[(r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \right] \right\rangle & \text{if } i = p \\ \text{unknown} & \text{if } i = p-1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\Omega := \omega_1 \wedge \omega_2 \wedge ... \wedge \omega_p$.

Theorem 1.4. *The cohomology groups have $\mathfrak{sl}(2)-$module structure:*

1. $$\bigoplus_{k=0}^{\infty}\left\langle\left[(r^2)^k\cdot\sum_{\alpha,\beta=1}^{p} z_\alpha z_\beta\otimes\omega_\alpha\otimes e_\beta\right]\right\rangle\simeq([\,\circ\,).$$

2. $$\bigoplus_{l=0}^{\infty}\left\langle\left[z_{p+1}^{2l+1}\cdot\sum_{\alpha=1}^{p} z_\alpha\otimes\Omega\otimes e_\alpha\right]\right\rangle\simeq\begin{cases}(\,\circ\,]\,\circ\,]) & \textit{if } p \textit{ is odd.}\\ (\,\circ\,]) & \textit{if } p \textit{ is even.}\end{cases}$$

3. $$\bigoplus_{l=0}^{\infty}\left\langle\left[z_{p+1}^{2l}\cdot\sum_{\alpha=1}^{p} z_\alpha\otimes\Omega\otimes e_\alpha\right]\right\rangle\bigoplus\bigoplus_{k=1}^{\infty}\left\langle\left[(r^2)^k\cdot\sum_{\alpha=1}^{p} z_\alpha\otimes\Omega\otimes e_\alpha\right]\right\rangle$$
$$\simeq\begin{cases}(\,\circ\,) & \textit{f } p \textit{ is odd.}\\ (\,\circ\,]\,\circ\,) & \textit{if } p \textit{ is even.}\end{cases}$$

Finally, we will prove the following general result:

Theorem 1.5. *For all $l\geq 1$, and all $3\leq i\leq p-3$ we have:*

$$H^i\left(\mathfrak{g},K;\mathcal{F}\otimes\mathrm{Sym}^l(V)\right)=\{0\}.$$

Overview

In Section 2 we will define Lie algebra cohomology using the Chevalley-Eilenberg complex and introduce the Fock model of the Weil representation, $(\omega,\mathcal{F})$, of $\mathfrak{o}(p,1)\times\mathfrak{sl}(2)$. We will then discuss the complexes with coefficients involving the Weil representation which will be of interest to us in this report.

In Section 3 we will describe a structure theorem for $\mathfrak{sl}(2)-$modules which satisfy certain properties. We will then observe in Section 5 that the cohomology groups computed in this report are in fact $\mathfrak{sl}(2)-$modules which satisfy these properties. The theorem given in this section will therefore allow us to easily describe the $\mathfrak{sl}(2)-$module structure of the cohomology groups.

Section 4 will see us develop the necessary weight theory for $\mathfrak{so}(m)$ representations to state a decomposition rule for the tensor product of irreducible $\mathfrak{so}(m)$ representations. In particular, we will discuss the decomposition of $\mathrm{Sym}^d(\mathbb{C}^m)$ into irreducible representations and provide an explicit formula for the decomposition of $\mathrm{Sym}^a(\mathbb{C}^m)\otimes\mathrm{Sym}^b(\mathbb{C}^m)$ into irreducible representations. We will use this result to calculate dimensions of subspaces of $C_l^i=\left[\mathcal{F}\otimes\bigwedge^i\mathfrak{p}^*\otimes\mathrm{Sym}^l(V)\right]^K$, which appear in Section 5.

In Section 5 we will calculate the relative Lie algebra cohomology of the Weil representation with coefficients in $\mathrm{Sym}^l(V)$, that is, the cohomology associated to the complexes $C_l^\bullet=\left[\mathcal{F}\otimes\bigwedge^\bullet\mathfrak{p}^*\otimes\mathrm{Sym}^l(V)\right]^K$ for $l=0,1$. We will give a complete description of all cohomology groups for $l=0$ and an almost complete description in the case $l=1$. We will observe that the cohomology groups are $\mathfrak{sl}(2)-$modules, and we will describe the structure of these modules. Finally, we will briefly discuss the general case of $l\in\mathbb{Z}_{\geq 0}$, in particular, we will show that the cohomology groups are $\{0\}$ in the case that $3\leq i\leq p-3$ for all $l\in\mathbb{Z}_{\geq 0}$.

2 Relative Lie algebra cohomology and the Weil representation

2.1 What is relative Lie algebra cohomology?

This subsection follows [2, Chapter 1.1]. Given a Lie algebra $\mathfrak{g}$ over a field k (usually $k = \mathbb{R}$ or $\mathbb{C}$) and a representation V of $\mathfrak{g}$, we can form a sequence of vector spaces

$$C^\bullet := C^\bullet(\mathfrak{g}; V) := \left[V \otimes \bigwedge^\bullet \mathfrak{g}^*\right] \cong \mathrm{Hom}_k\left(\bigwedge^\bullet \mathfrak{g}, V\right).$$

We have linear maps

$$d^i : C^i = \mathrm{Hom}_k\left(\bigwedge^i \mathfrak{g}, V\right) \longrightarrow C^{i+1} = \mathrm{Hom}_k\left(\bigwedge^{i+1} \mathfrak{g}, V\right)$$

given by

$$\begin{aligned}(d^i f)(x_1, ..., x_{i+1}) &= \sum_{k=1}^{i+1}(-1)^k x_k \cdot f(x_1, ..., \widehat{x_k}, ..., x_i)\\ &+ \sum_{k<l}(-1)^{k+l} f([x_k, x_l], x_1, ..., \widehat{x_k}, ..., \widehat{x_l}, ..., x_i),\end{aligned}$$

where $\widehat{x_k}$ denotes the omission of the entry x_k and where $[x_k, x_l]$ denotes the Lie bracket of x_k and x_l.

Lemma 2.1. *$d^2 := d^{i+1} \circ d^i : C^i \longrightarrow C^{i+2} \equiv 0$ for all $i \in \mathbb{Z}_{\geq 0}$.*

Proof. See [5, Chapter IV]. □

Thus, $(C^\bullet(\mathfrak{g}; V), d)$ is a co-chain complex. This complex is commonly referred to as the Chevalley-Eilenburg complex and the linear maps, d, are referred to as differentials. We define the i^{th} cohomology group of this complex as follows:

Definition 2.2. The i^{th} cohomology group, denoted $H^i(\mathfrak{g}; V)$, is the quotient group

$$H^i(\mathfrak{g}; V) = {\ker\left(d^i : C^i \longrightarrow C^{i+1}\right)} \Big/ {\mathrm{im}\left(d^{i-1} : C^{i-1} \longrightarrow C^i\right)}.$$

This cohomology is known as the Lie algebra cohomology of $\mathfrak{g}$ with coefficients in V.

Furthermore, given a Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ we consider the sequence of spaces

$$C^\bullet(\mathfrak{g}, \mathfrak{k}; V) := \left[V \otimes \bigwedge^\bullet (\mathfrak{g}/\mathfrak{k})^*\right]^{\mathfrak{k}} \cong \mathrm{Hom}_{\mathfrak{k}}\left(\bigwedge^\bullet (\mathfrak{g}/\mathfrak{k}), V\right).$$

The above differentials d^i restrict to differentials $d^i : C^i(\mathfrak{g}, \mathfrak{k}; V) \longrightarrow C^{i+1}(\mathfrak{g}, \mathfrak{k}; V)$, and so we obtain a new co-chain complex $(C^\bullet(\mathfrak{g}, \mathfrak{k}; V), d)$. The i^{th} cohomology group of this co-chain complex is given as follows:

Definition 2.3.

$$H^i\big(\mathfrak{g},\mathfrak{k};V\big) = {\ker\Big(d^i : C^i(\mathfrak{g},\mathfrak{k};V) \to C^{i+1}(\mathfrak{g},\mathfrak{k};V)\Big)}\Big/{\operatorname{im}\big(d^{i-1} : C^{i-1}(\mathfrak{g},\mathfrak{k};V) \to C^i(\mathfrak{g},\mathfrak{k};V)\big)}.$$

This cohomology is known as the Lie algebra cohomology of $\mathfrak{g}$ relative to $\mathfrak{k}$ with coefficients in V.

Now, let G be a Lie group with $\mathfrak{g} = \mathrm{Lie}(G)$, and let K be the maximal compact subgroup of G with $\mathfrak{k} = \mathrm{Lie}(K)$. We define

$$C^\bullet(\mathfrak{g},K;V) := \mathrm{Hom}_K\left(\bigwedge\nolimits^\bullet (\mathfrak{g}/\mathfrak{k}), V\right).$$

If K is connected we have [2, Chapter I, Section 5.1]:

$$C^\bullet(\mathfrak{g},\mathfrak{k};V) = C^\bullet(\mathfrak{g},K;V)$$

and

$$H^\bullet(\mathfrak{g},\mathfrak{k};V) = H^\bullet(\mathfrak{g},K;V).$$

2.2 The Weil representation of $\mathfrak{o}(p,1)\times\mathfrak{sl}(2)$

In this subsection we will introduce the Fock model of the Weil representation, $(\omega,\mathcal{F})$, of the reductive dual pair $\mathfrak{o}(p,1)\times\mathfrak{sl}(2)$. We follow [6, Appendix A] and [4, Chapter 7].

Remark 2.4. Throughout this report we will use "early Greek letters", e.g. α,β,γ, to denote values from 1 to p, unless stated otherwise.

Let $\big(V,(\cdot,\cdot)\big)$ be a real quadratic space of signature $(p,1)$ with orthonormal basis $\{e_1,e_2,...,e_p,e_{p+1}\}$, that is:

$$(e_\alpha,e_\alpha) = 1 \text{ for } 1\le\alpha\le p, \quad (e_{p+1},e_{p+1}) = -1.$$

Then,

$$\begin{aligned}\mathfrak{o}(V) = \mathfrak{o}(p,1) &= \big\{X\in\mathfrak{gl}(p+1,\mathbb{R}) \;\big|\; (Xv,w)+(v,Xw)=0 \text{ for all } v,w\in V\big\}\\ &= \big\{X\in\mathfrak{gl}(p+1,\mathbb{R}) \;\big|\; X^T I_{p,1} + I_{p,1}X = 0\big\}.\end{aligned}$$

where $I_{p,1} = \mathrm{diag}(\underbrace{1,...,1}_{\text{p times}},-1)$. Let $E_{p,q}$ denote the matrix with a 1 in entry (p,q) and 0s elsewhere. An explicit calculation shows that

$$\mathfrak{o}(p,1) = \langle X_{\alpha,\beta}\,,\,X_{\gamma,p+1}\rangle_{\mathbb{R}}, \quad 1\le\alpha<\beta\le p, \quad 1\le\gamma\le p,$$

where $X_{\alpha,\beta} = -E_{\alpha,\beta}+E_{\beta,\alpha}$, $X_{\gamma,p+1} = E_{\gamma,p+1}+E_{p+1,\gamma}$, and $\langle\,\cdot\,\rangle_{\mathbb{R}}$ denotes the real span.

Example 2.5. Let $p = 3$, then examples of matrices in $\mathfrak{o}(3,1)$ are:

$$X_{1,2} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_{2,4} = X_{2,p+1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Lemma 2.6. *Let $\theta(X) = -X^T$, then we have the following eigenspace decomposition with respect to θ:*

$$\mathfrak{o}(p,1) = \mathfrak{k} \oplus \mathfrak{p}_0$$

where

$$\mathfrak{k} = \langle X_{\alpha,\beta} \mid 1 \leq \alpha < \beta \leq p \rangle \text{ is the } +1 \text{ eigenspace}$$

and

$$\mathfrak{p}_0 = \langle X_{\gamma,p+1} \mid 1 \leq \gamma \leq p \rangle \text{ is the } -1 \text{ eigenspace.}$$

Now, let $\mathfrak{sl}(2,\mathbb{C})$ have basis

$$H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad R = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad L = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

We have Lie bracket relations $[R,L] = -iH$, $[H,R] = 2iR$, $[H,L] = -2iL$. Let $\mathcal{F} = \mathbb{C}[z_1, z_2, ...z_{p+1}]$ be the space of complex polynomials in variables $z_1, z_2, ..., z_{p+1}$. Then we have the following actions of $\mathfrak{o}(p,1)$ and $\mathfrak{sl}(2;\mathbb{C})$ on $\mathcal{F}$, where the formulae are given, up to scaling, as in [6, Lemma A.1, Lemma A.2] after applying the change of variable $z_i \mapsto \sqrt{2}z_i$ (for convenience) and letting $\lambda = i$.

Definition 2.7. The Weil representation of $\mathfrak{o}(p,1) \times \mathfrak{sl}(2,\mathbb{C})$, denoted $(\omega, \mathcal{F})$, is given by the following actions on $\mathcal{F}$:

1. The action of $\mathfrak{o}(p,1)$ on $\mathcal{F}$ is given by:

$$\begin{aligned} \omega(X_{\alpha,\beta}) &= -z_\alpha \frac{\partial}{\partial z_\beta} + z_\beta \frac{\partial}{\partial z_\alpha} \\ \omega(X_{\alpha,p+1}) &= -\frac{\partial^2}{\partial z_\alpha \partial z_{p+1}} + z_\alpha z_{p+1}. \end{aligned}$$

2. The action of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathcal{F}$ is given by:

$$\begin{aligned} \omega(H) &= i\left[\sum_{\alpha=1}^{p} z_\alpha \frac{\partial}{\partial z_\alpha} - z_{p+1}\frac{\partial}{\partial z_{p+1}}\right] + \frac{(p-1)i}{2} \\ \omega(R) &= \frac{1}{2}\sum_{\alpha=1}^{p} z_\alpha^2 - \frac{1}{2}\frac{\partial^2}{\partial z_{p+1}^2} \\ \omega(L) &= -\frac{1}{2}\sum_{\alpha=1}^{p}\frac{\partial^2}{\partial z_\alpha^2} + \frac{1}{2}z_{p+1}^2. \end{aligned}$$

Proposition 2.8. *$(\omega,\mathcal{F})$ is a representation of $\mathfrak{o}(p,1)\times\mathfrak{sl}(2,\mathbb{C})$.*

Proof. There are three things to show, namely:

1. $\omega\big([X,Y]\big)=\big[\omega(X),\omega(Y)\big]$ for all $X,Y\in\mathfrak{o}(p,1)$.
2. $\omega\big([X,Y]\big)=\big[\omega(X),\omega(Y)\big]$ for all $X,Y\in\mathfrak{sl}(2,\mathbb{C})$.
3. The actions of $\mathfrak{o}(p,1)$ and $\mathfrak{sl}(2,\mathbb{C})$ commute. That is: $\omega(X)\omega(Y)=\omega(Y)\omega(X)$ for all $X\in\mathfrak{o}(p,1),Y\in\mathfrak{sl}(2,\mathbb{C})$.

As both the Lie bracket and ω are linear, we only need to prove the relations for the basis elements of $\mathfrak{o}(p,1)$ and $\mathfrak{sl}(2,\mathbb{C})$.

1. Let $X_{\alpha,\beta}\in\mathfrak{k}$, $X_{\gamma,\delta}\in\mathfrak{k}$, then we have:

$$\begin{aligned}
[X_{\alpha,\beta},X_{\gamma,\delta}] &= (X_{\alpha,\beta})\cdot(X_{\gamma,\delta})-(X_{\gamma,\delta})\cdot(X_{\alpha,\beta})\\
&= (-E_{\alpha,\beta}+E_{\beta,\alpha})\cdot(-E_{\gamma,\delta}+E_{\delta,\gamma})-(-E_{\gamma,\delta}+E_{\delta,\gamma})\cdot(-E_{\alpha,\beta}+E_{\beta,\alpha})\\
&= \big(E_{\alpha,\delta}\delta_{\beta,\gamma}-E_{\alpha,\gamma}\delta_{\beta,\delta}-E_{\beta,\delta}\delta_{\alpha,\gamma}+E_{\beta,\gamma}\delta_{\alpha,\delta}\big)\\
&- \big(E_{\gamma,\beta}\delta_{\alpha,\delta}-E_{\gamma,\alpha}\delta_{\delta,\beta}-E_{\delta,\beta}\delta_{\gamma,\alpha}+E_{\delta,\alpha}\delta_{\gamma,\beta}\big)\\
&= X_{\alpha,\gamma}\delta_{\beta,\delta}+X_{\delta,\alpha}\delta_{\gamma,\beta}+X_{\gamma,\beta}\delta_{\alpha,\delta}+X_{\beta,\delta}\delta_{\alpha,\gamma}
\end{aligned}$$

$$\implies \omega\big([X_{\alpha,\beta},X_{\gamma,\delta}]\big)=\omega(X_{\alpha,\gamma})\delta_{\beta,\delta}+\omega(X_{\delta,\alpha})\delta_{\gamma,\beta}+\omega(X_{\gamma,\beta})\delta_{\alpha,\delta}+\omega(X_{\beta,\delta})\delta_{\alpha,\gamma}.$$

On the other hand,

$$\begin{aligned}
\omega(X_{\alpha,\beta})\cdot\omega(X_{\gamma,\delta}) &= \Big(-z_\alpha\frac{\partial}{\partial z_\beta}+z_\beta\frac{\partial}{\partial z_\alpha}\Big)\cdot\Big(-z_\gamma\frac{\partial}{\partial z_\delta}+z_\delta\frac{\partial}{\partial z_\gamma}\Big)\\
&= z_\alpha\frac{\partial}{\partial z_\delta}\delta_{\beta,\gamma}+z_\alpha z_\gamma\frac{\partial^2}{\partial z_\beta\partial z_\delta}-z_\alpha\frac{\partial}{\partial z_\gamma}\delta_{\beta,\delta}-z_\alpha z_\delta\frac{\partial^2}{\partial z_\beta\partial z_\gamma}\\
&- z_\beta\frac{\partial}{\partial z_\delta}\delta_{\alpha,\gamma}-z_\beta z_\gamma\frac{\partial^2}{\partial z_\alpha\partial z_\delta}+z_\beta\frac{\partial}{\partial z_\gamma}\delta_{\alpha,\delta}+z_\beta z_\delta\frac{\partial^2}{\partial z_\alpha\partial z_\gamma}
\end{aligned}$$

$$\begin{aligned}
\omega(X_{\gamma,\delta})\cdot\omega(X_{\alpha,\beta}) &= \Big(-z_\gamma\frac{\partial}{\partial z_\delta}+z_\delta\frac{\partial}{\partial z_\gamma}\Big)\cdot\Big(-z_\alpha\frac{\partial}{\partial z_\beta}+z_\beta\frac{\partial}{\partial z_\alpha}\Big)\\
&= z_\gamma\frac{\partial}{\partial z_\beta}\delta_{\delta,\alpha}+z_\gamma z_\alpha\frac{\partial^2}{\partial z_\delta\partial z_\beta}-z_\gamma\frac{\partial}{\partial z_\alpha}\delta_{\delta,\beta}-z_\gamma z_\beta\frac{\partial^2}{\partial z_\delta\partial z_\alpha}\\
&- z_\delta\frac{\partial}{\partial z_\beta}\delta_{\gamma,\alpha}-z_\delta z_\alpha\frac{\partial^2}{\partial z_\gamma\partial z_\beta}+z_\delta\frac{\partial}{\partial z_\alpha}\delta_{\gamma,\beta}+z_\delta z_\beta\frac{\partial^2}{\partial z_\gamma\partial z_\alpha}
\end{aligned}$$

Thus,

$$\begin{aligned}
\big[\omega(X_{\alpha,\beta}),\omega(X_{\gamma,\delta})\big] &= \Big(-z_\alpha\frac{\partial}{\partial z_\gamma}+z_\gamma\frac{\partial}{\partial z_\alpha}\Big)\delta_{\beta,\delta}+\Big(-z_\delta\frac{\partial}{\partial z_\alpha}+z_\alpha\frac{\partial}{\partial z_\delta}\Big)\delta_{\beta,\gamma}\\
&+ \Big(-z_\gamma\frac{\partial}{\partial z_\beta}+z_\beta\frac{\partial}{\partial z_\gamma}\Big)\delta_{\alpha,\delta}+\Big(-z_\beta\frac{\partial}{\partial z_\delta}+z_\delta\frac{\partial}{\partial z_\beta}\Big)\delta_{\alpha,\gamma}\\
&= \omega\big([X_{\alpha,\beta},X_{\gamma,\delta}]\big)
\end{aligned}$$

Similarly, one can show the relation holds for $X_{\alpha,\beta} \in \mathfrak{k}$, $X_{\alpha,p+1} \in \mathfrak{p}_0$ and also for $X_{\alpha,p+1} \in \mathfrak{p}_0$, $X_{\beta,p+1} \in \mathfrak{p}_0$.

2. We will show that $\omega\big([R,L]\big) = \big[\omega(R), \omega(L)\big]$, other relations can be similarly shown.

$$\begin{aligned}\omega(R)\cdot\omega(L) &= \frac{1}{4}\Bigg(\Bigg(\sum_{\alpha=1}^{p} z_\alpha^2 - \frac{\partial^2}{\partial z_{p+1}^2}\Bigg)\cdot\Bigg(-\sum_{\beta=1}^{p}\frac{\partial^2}{\partial z_\beta^2} + z_{p+1}^2\Bigg)\Bigg)\\ &= \frac{1}{4}\Bigg(-\sum_{\alpha,\beta=1}^{p} z_\alpha^2\frac{\partial^2}{\partial z_\beta^2} + \sum_{\alpha=1}^{p} z_\alpha^2 z_{p+1}^2 + \sum_{\beta=1}^{p}\frac{\partial^4}{\partial z_{p+1}^2\partial z_\beta^2} - \frac{\partial^2}{\partial z_{p+1}^2} z_{p+1}^2\Bigg).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\omega(L)\cdot\omega(R) &= \frac{1}{4}\Bigg(\Bigg(-\sum_{\beta=1}^{p}\frac{\partial^2}{\partial z_\beta^2} + z_{p+1}^2\Bigg)\cdot\Bigg(\sum_{\alpha=1}^{p} z_\alpha^2 - \frac{\partial^2}{\partial z_{p+1}^2}\Bigg)\Bigg)\\ &= \frac{1}{4}\Bigg(-\sum_{\alpha,\beta=1}^{p}\frac{\partial^2}{\partial z_\beta^2} z_\alpha^2 + \sum_{\beta=1}^{p}\frac{\partial^4}{\partial z_\beta^2\partial z_{p+1}^2} + \sum_{\alpha=1}^{p} z_{p+1}^2 z_\alpha^2 - z_{p+1}^2\frac{\partial^2}{\partial z_{p+1}^2}\Bigg).\end{aligned}$$

Now, we have the relation

$$\frac{\partial^2}{\partial z_\beta^2} z_\alpha^2 = 2\delta_{\alpha,\beta} + 4z_\alpha\delta_{\alpha,\beta}\frac{\partial}{\partial z_\beta} + z_\alpha^2\frac{\partial^2}{\partial z_\beta^2},$$

thus, we obtain

$$\begin{aligned}\big[\omega(R),\omega(L)\big] &= \omega(R)\cdot\omega(L) - \omega(L)\cdot\omega(R)\\ &= \frac{1}{4}\Bigg(\sum_{\alpha=1}^{p}\Bigg(2 + 4z_\alpha\frac{\partial}{\partial z_\alpha}\Bigg) - 2 - 4z_{p+1}\frac{\partial}{\partial z_{p+1}}\Bigg)\\ &= \sum_{\alpha=1}^{p} z_\alpha\frac{\partial}{\partial z_\alpha} - z_{p+1}\frac{\partial}{\partial z_{p+1}} + \frac{p-1}{2}\\ &= -i\Bigg(i\Bigg[\sum_{\alpha=1}^{p} z_\alpha\frac{\partial}{\partial z_\alpha} - z_{p+1}\frac{\partial}{\partial z_{p+1}}\Bigg] + \frac{(p-1)i}{2}\Bigg)\\ &= -i\,\omega(H) = \omega(-iH)\\ &= \omega\big([R,L]\big).\end{aligned}$$

3. We will show that the action of $X_{\alpha,\beta} \in \mathfrak{o}(p,1)$ and $H \in \mathfrak{sl}(2,\mathbb{C})$ commute. The

relation can be similarly shown for all other pairs.

$$
\begin{aligned}
&\omega(X_{\alpha,\beta})\cdot\omega(H)\\
&= i\left(-z_\alpha\frac{\partial}{\partial z_\beta}+z_\beta\frac{\partial}{\partial z_\alpha}\right)\cdot\left(\sum_{\gamma=1}^{p} z_\gamma\frac{\partial}{\partial z_\gamma}-z_{p+1}\frac{\partial}{\partial z_{p+1}}\right)+\frac{(p-1)i}{2}\omega(X_{\alpha,\beta})\\
&= i\left(\sum_{\gamma=1}^{p}\left(-z_\alpha\frac{\partial}{\partial z_\beta}z_\gamma\frac{\partial}{\partial z_\gamma}+z_\beta\frac{\partial}{\partial z_\alpha}z_\gamma\frac{\partial}{\partial z_\gamma}\right)+z_\alpha\frac{\partial}{\partial z_\beta}z_{p+1}\frac{\partial}{\partial z_{p+1}}-z_\beta\frac{\partial}{\partial z_\alpha}z_{p+1}\frac{\partial}{\partial z_{p+1}}\right)\\
&\quad+\frac{(p-1)i}{2}\omega(X_{\alpha,\beta})\\
&= i\left(-z_\alpha\frac{\partial}{\partial z_\beta}-\sum_{\gamma=1}^{p} z_\alpha z_\gamma\frac{\partial^2}{\partial z_\beta\partial z_\gamma}+z_\beta\frac{\partial}{\partial z_\alpha}+\sum_{\gamma=1}^{p} z_\beta z_\gamma\frac{\partial^2}{\partial z_\alpha\partial z_\gamma}\right.\\
&\quad\left.+z_\alpha z_{p+1}\frac{\partial^2}{\partial z_\beta\partial z_{p+1}}-z_\beta z_{p+1}\frac{\partial^2}{\partial z_\alpha\partial z_{p+1}}\right)+\frac{(p-1)i}{2}\omega(X_{\alpha,\beta}).
\end{aligned}
$$

$$
\begin{aligned}
&\omega(H)\cdot\omega(X_{\alpha,\beta})\\
&= i\left(\sum_{\gamma=1}^{p} z_\gamma\frac{\partial}{\partial z_\gamma}-z_{p+1}\frac{\partial}{\partial z_{p+1}}\right)\cdot\left(-z_\alpha\frac{\partial}{\partial z_\beta}+z_\beta\frac{\partial}{\partial z_\alpha}\right)+\frac{(p-1)i}{2}\omega(X_{\alpha,\beta})\\
&= i\left(\sum_{\gamma=1}^{p}\left(-z_\gamma\frac{\partial}{\partial z_\gamma}z_\alpha\frac{\partial}{\partial z_\beta}+z_\gamma\frac{\partial}{\partial z_\gamma}z_\beta\frac{\partial}{\partial z_\alpha}\right)+z_{p+1}\frac{\partial}{\partial z_{p+1}}z_\alpha\frac{\partial}{\partial z_\beta}-z_{p+1}\frac{\partial}{\partial z_{p+1}}z_\beta\frac{\partial}{\partial z_\alpha}\right)\\
&\quad+\frac{(p-1)i}{2}\omega(X_{\alpha,\beta})\\
&= i\left(-z_\alpha\frac{\partial}{\partial z_\beta}-\sum_{\gamma=1}^{p} z_\gamma z_\alpha\frac{\partial^2}{\partial z_\gamma\partial z_\beta}+z_\beta\frac{\partial}{\partial z_\alpha}+\sum_{\gamma=1}^{p} z_\gamma z_\beta\frac{\partial^2}{\partial z_\gamma\partial z_\alpha}\right.\\
&\quad\left.+z_{p+1}z_\alpha\frac{\partial^2}{\partial z_{p+1}\partial z_\beta}-z_{p+1}z_\beta\frac{\partial^2}{\partial z_{p+1}\partial z_\alpha}\right)+\frac{(p-1)i}{2}\omega(X_{\alpha,\beta}).
\end{aligned}
$$

From which one observes that indeed $\omega(X_{\alpha,\beta})\cdot\omega(H)=\omega(H)\cdot\omega(X_{\alpha,\beta})$.

$\square$

2.3 Complexes with coefficients involving the Weil representation

We will now introduce the complexes of interest in this report. Let $G=SO_0(p,1)$, the connected component to the identity of $O(p,1)$. Let $K=SO(p)\times SO(1)$, the maximal compact subgroup of G. Then $\mathfrak{g}=\mathrm{Lie}(G)=\mathfrak{o}(p,1)$, $\mathfrak{k}=\mathrm{Lie}(K)=\mathfrak{so}(p)\oplus\mathfrak{so}(1)\cong\mathfrak{so}(p)$. Recall that we have the decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}_0$, and also recall that $\mathfrak{p}_0$ has basis $\{X_{\alpha,p+1}\}_{\alpha=1}^{p}$. Let $\mathfrak{p}_0^*$ have canonical basis $\{\omega_\alpha\}_{\alpha=1}^{p}$ such that $\omega_\alpha(X_{\beta,p+1})=\delta_{\alpha\beta}$.

Consider the sequence of spaces

$$C^\bullet(\mathfrak{g}, K; \mathcal{F}) = \left[\mathcal{F} \otimes \bigwedge\nolimits^\bullet \left({}^{\mathfrak{g}}\!/\!_{\mathfrak{k}}\right)^*\right]^K = \left[\mathcal{F} \otimes \bigwedge\nolimits^\bullet \mathfrak{p}_0^*\right]^K =^1 \left[\mathcal{F} \otimes \bigwedge\nolimits^\bullet \mathfrak{p}_0^*\right]^{\mathfrak{k}}$$

with differential given by [3, Theorem 2.1.3]

$$d = \sum_{\alpha=1}^{p} \omega(X_{\alpha,p+1}) \otimes A(\omega_\alpha) = \sum_{\alpha=1}^{p} \left(-\frac{\partial^2}{\partial z_\alpha \partial z_{p+1}} + z_\alpha z_{p+1}\right) \otimes A(\omega_\alpha),$$

where $A(\omega_\alpha)$ denotes left exterior multiplication on $\bigwedge^i \mathfrak{p}_0^*$ by $\omega_\alpha \in \mathfrak{p}_0^*$. This complex will enable us to compute the Lie algebra cohomology of $\mathfrak{g}$ relative to $\mathfrak{k}$ (or K) with coefficients in the Weil representation, otherwise known as the relative Lie algebra cohomology of the Weil representation with trivial coefficients.

Furthermore, if we let V denote the standard representation of $SO_0(p,1)$, then we may consider the sequence of spaces

$$C^\bullet\left(\mathfrak{g}, K; \mathcal{F} \otimes \mathrm{Sym}^l(V)\right) = \left[\mathcal{F} \otimes \bigwedge\nolimits^\bullet \mathfrak{p}_0^* \otimes \mathrm{Sym}^l(V)\right]^K.$$

In this case the differential is given by [6, Section 5.1]

$$d = d_S + d_V$$

where

$$d_S = \sum_{\alpha=1}^{p} \omega(X_{\alpha,p+1}) \otimes A(\omega_\alpha) \otimes 1, \quad d_V = \sum_{\alpha=1}^{p} 1 \otimes A(\omega_\alpha) \otimes D\rho(X_{\alpha,p+1})$$

where $D\rho$ is the derived action of $\mathfrak{p} \subset \mathfrak{g} = \mathrm{Lie}(G)$ on $\mathrm{Sym}^l(V)$. This complex will allow us to compute the Lie algebra cohomology of $\mathfrak{g}$ relative to $\mathfrak{k}$ (or K) with coefficients in $\mathcal{F} \otimes \mathrm{Sym}^l(V)$, otherwise known as the relative Lie algebra cohomology of the Weil representation with coefficients in $\mathrm{Sym}^l(V)$.

Remark 2.9. Observe that for $l = 0$, $d_V \equiv 0$ and the latter complex reduces to the former.

Now let $\mathfrak{p}$ denote the complexification of $\mathfrak{p}_0$, that is, $\mathfrak{p} = \mathfrak{p}_0 \otimes \mathbb{C}$. Also let $K_\mathbb{C} = SO(p, \mathbb{C}) \times SO(1, \mathbb{C})$. We have the following isomorphism [3, Chapter 2.2]

$$C^\bullet(\mathfrak{g}, K; \mathcal{F}) \cong C^\bullet\big(\mathfrak{o}(p,1,\mathbb{C}), K_\mathbb{C}, \mathcal{F}\big).$$

This extends to an isomorphism

$$C^\bullet\left(\mathfrak{g}, K; \mathcal{F} \otimes \mathrm{Sym}^l(V)\right) \cong C^\bullet\left(\mathfrak{o}(p,1,\mathbb{C}), K_\mathbb{C}; \mathcal{F} \otimes \mathrm{Sym}^l(V \otimes \mathbb{C})\right)$$

where

$$C^\bullet\big(\mathfrak{o}(p,1,\mathbb{C}), K_\mathbb{C}, \mathcal{F} \otimes \mathrm{Sym}^l(V \otimes \mathbb{C})\big) = \left[\mathcal{F} \otimes \bigwedge\nolimits^\bullet \mathfrak{p}^* \otimes \mathrm{Sym}^l(V \otimes \mathbb{C})\right]^{SO(p,\mathbb{C})}.$$

We will make use of this isomorphism in Section 5.

[1] As K is connected these spaces are equal, see [2, Chapter I, Section 5.1].

3 A structure theorem for $\mathfrak{sl}(2)-$modules

We will see in Section 5 that the cohomology groups arising from the complexes

$$C^i\big(\mathfrak{o}(p,1,\mathbb{C}), K_{\mathbb{C}}, \mathcal{F} \otimes \mathrm{Sym}^l(V \otimes \mathbb{C})\big)$$

are in fact $\mathfrak{sl}(2)-$modules. These modules will be infinite dimensional and so in order to classify them we will state and describe a structure theorem for $\mathfrak{sl}(2)-$modules. This section will follow [7, Section II, Chapter 1]. We will discuss the result as it is found in [7], that is, a result about $\mathfrak{sl}(2,\mathbb{R})-$modules. We will then extend this result to a result about $\mathfrak{sl}(2,\mathbb{C})-$modules using complexification. Before we state the theorem we will introduce some terminology.

3.1 Definitions and terminology

We begin by remarking that the definitions and terminology in this section apply to both $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sl}(2,\mathbb{C})$, therefore we will simply write $\mathfrak{sl}(2)$. We will also use the terms "$\mathfrak{sl}(2)-$module" and "representation of $\mathfrak{sl}(2)$" interchangeably throughout the section (and throughout the report).

Definition 3.1 (Standard basis of $\mathfrak{sl}(2)$)**.** Let $\mathfrak{sl}(2)$ have basis $\{h, e, f\}$. We say $\{h, e, f\}$ is a standard basis of $\mathfrak{sl}(2)$ if it satisfies the following Lie bracket relations:

$$[f, e] = h, \quad [h, e] = -2e, \quad [h, f] = 2f.$$

Definition 3.2 (h-semisimple module)**.** Let (ρ, V) be a representation of $\mathfrak{sl}(2)$. If V has a decomposition

$$V = \bigoplus_{\lambda} V_\lambda$$

where

$$V_\lambda = \{v \in V : \rho(h)v = \lambda v\}$$

and each V_λ is finite-dimensional then we say (ρ, V) is h-semisimple. We say λ are the weights of the representation V; the $v \in V_\lambda$ are weight vectors corresponding to the weight λ; and the V_λ are weight spaces.

Example 3.3. Consider $\mathfrak{sl}(2,\mathbb{R})$ with standard basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; e = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and let $(\rho, V) = (\rho, \mathbb{R}^2)$ be the standard representation of $\mathfrak{sl}(2)$, where $\mathbb{R}^2$ has standard basis $\{e_1, e_2\}$. Then

$$V = V_1 \oplus V_{-1}$$

where

$$V_1 = \{v \in V : \rho(h)v = v\} = \mathbb{R} \cdot e_1$$

and

$$V_{-1} = \{v \in V : \rho(h)v = -v\} = \mathbb{R} \cdot e_2.$$

Definition 3.4 (h multiplicity free $\mathfrak{sl}(2)-$module)**.** Let (ρ, V) be a representation of $\mathfrak{sl}(2)$ and suppose V is h-semisimple, that is,

$$V = \bigoplus_{\lambda} V_\lambda$$

where $V_\lambda = \{v \in V : \rho(h)v = \lambda v\}$. Then we say V is h multiplicity free if for all λ such that $V_\lambda \neq \{0\}$ we have

$$\dim(V_\lambda) = 1.$$

Definition 3.5 (Indecomposable $\mathfrak{sl}(2)-$module)**.** Let (ρ, V) be a representation of $\mathfrak{sl}(2)$. We say V is indecomposable if $V \neq \{0\}$ and V cannot be expressed as a direct sum, $V = U_1 \oplus U_2$, of two non-zero subrepresentations of V.

Remark 3.6. If V is irreducible then V is indecomposable. The converse is not true.

Definition 3.7 (Casimir operator and quasisimple $\mathfrak{sl}(2)-$modules)**.** Let $\mathfrak{sl}(2)$ have standard basis $\{h, e, f\}$, then we define the Casimir operator as

$$\mathcal{C} = h^2 + 2ef + 2fe.$$

Furthermore, let (ρ, V) be a representation of $\mathfrak{sl}(2)$, then we say that V is quasisimple if $\mathcal{C}$ acts as a multiple of the identity on V, i.e.

$$\mathcal{C} \cdot v = \big(\rho(h)\rho(h) + 2\rho(e)\rho(f) + 2\rho(f)\rho(e)\big) \cdot v = \lambda v$$

for all $v \in V$, where λ is a scalar depending on v.

Proposition 3.8. *[7, Proposition 1.1.4] Let (ρ, V) be a representation of $\mathfrak{sl}(2)$ and let $\mathfrak{sl}(2)$ have standard basis $\{h, e, f\}$. If $v \in V$ is a weight vector with weight λ, then:*

1. *$\rho(f)^k v$, $k \in \mathbb{Z}_{\geq 0}$ is either zero or a weight vector with weight $\lambda + 2k$.*

2. *$\rho(e)^k v$, $k \in \mathbb{Z}_{\geq 0}$ is either zero or a weight vector with weight $\lambda - 2k$.*

Thus, $\mathfrak{sl}(2) \cdot v \subseteq V$, the $\mathfrak{sl}(2)-$module generated by v, is h-semisimple and h multiplicity free.

Furthermore, suppose that v is an eigenvalue for $\mathcal{C}$, that is

$$\mathcal{C} \cdot v = \mu v$$

for some scalar μ. Then $\mathfrak{sl}(2) \cdot v$ is also quasisimple.

Definition 3.9. If $\{h, e, f\}$ is a standard basis for $\mathfrak{sl}(2)$ then we call e the lowering operator and f the raising operator (as they lower/raise the weight of weight vectors by ± 2). Furthermore, if $v \in V$ is a weight vector, i.e. $h \cdot v = \lambda \cdot v$ for some $\lambda \in \mathbb{C}$, then we say v is a highest weight vector if $f \cdot v = 0$. Similarly, we say v a lowest weight vector if $e \cdot v = 0$.

With these definitions we can now present a structure theorem for indecomposable, quasisimple, h-semisimple, h multiplicity free $\mathfrak{sl}(2)-$modules.

3.2 The structure theorem

The following theorem gives a pictorial way to view the isomorphism classes of indecomposable, quasisimple, h-semisimple, h multiplicity free $\mathfrak{sl}(2)-$modules.

The dots, $\circ$, on the diagrams below denote h-weight spaces ordered with increasing weights from left to right. A left bracket "[" indicates that the weight vectors in the weight space to the immediate right are killed by applying the lowering operator e, but that the weight vectors to the immediate left are not killed by applying the raising operator f. We analogously define the right bracket "]".

Theorem 3.10. *[7, Theorem 1.1.13] If V is an indecomposable, quasisimple, h-semisimple, h multiplicity free $\mathfrak{sl}(2)$-module then the isomorphism class of V is given by one of the following:*

$$
\begin{array}{lll}
(\circ) & V \simeq & \cdots \ \circ \ \circ \ \circ \ \circ \ \circ \ \circ \ \cdots \\
(\circ]\circ) & V \simeq & \cdots \ \circ \ \circ \ \cdots \ \circ \ \Big] \ \circ \ \circ \ \cdots \\
(\circ]) & V \simeq & \cdots \ \circ \ \circ \ \cdots \ \circ \ \Big] \\
(\circ[\circ) & V \simeq & \cdots \ \circ \ \circ \ \Big[\ \circ \ \cdots \ \circ \ \circ \ \cdots \\
([\circ) & V \simeq & \Big[\ \circ \ \cdots \ \circ \ \circ \ \cdots \\
(\circ[\circ]\circ) & V \simeq & \cdots \ \circ \ \Big[\ \circ \ \cdots \ \circ \ \Big] \ \circ \ \cdots \\
(\circ[\circ]) & V \simeq & \cdots \ \circ \ \Big[\ \circ \ \cdots \ \circ \ \Big] \\
([\circ]\circ) & V \simeq & \Big[\ \circ \ \cdots \ \circ \ \Big] \ \circ \ \cdots \\
([\circ]) & V \simeq & \Big[\ \circ \ \cdots \ \circ \ \Big] \\
(\circ]\circ]\circ) & V \simeq & \cdots \ \circ \ \Big] \ \circ \ \cdots \ \circ \ \Big] \ \circ \ \cdots \\
(\circ]\circ]) & V \simeq & \cdots \ \circ \ \Big] \ \circ \ \cdots \ \circ \ \Big] \\
(\circ[\circ[\circ) & V \simeq & \cdots \ \circ \ \Big[\ \circ \ \cdots \ \circ \ \Big[\ \circ \ \cdots \\
([\circ[\circ) & V \simeq & \Big[\ \circ \ \cdots \ \circ \ \Big[\ \circ \ \cdots \\
(\circ]\circ[\circ) & V \simeq & \cdots \ \circ \ \Big] \ \circ \ \cdots \ \circ \ \Big[\ \circ \ \cdots
\end{array}
$$

Remark 3.11. In [7], this theorem is given for $\mathfrak{sl}(2,\mathbb{R})-$modules. However, every $\mathbb{R}$-linear representation of $\mathfrak{sl}(2,\mathbb{R})$ extends uniquely to a $\mathbb{C}$-linear representation of $\mathfrak{sl}(2,\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C}=\mathfrak{sl}(2,\mathbb{C})$, so the theorem holds for $\mathbb{C}$-linear representations of $\mathfrak{sl}(2,\mathbb{C})$ also.

Recall that we stated this theorem in order to help us to describe the $\mathfrak{sl}(2)-$module structure of the cohomology groups which will arise in Section 5 of the report. However, much of the exposition in [7] (which has been omitted) leading up to the structure theorem uses that $\{h,e,f\}$ is a standard basis of $\mathfrak{sl}(2)$. Recall that in Section 2 we let $\mathfrak{sl}(2,\mathbb{C})$ have the basis

$$H=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\ R=\frac{1}{2}\begin{pmatrix}1&i\\i&-1\end{pmatrix},\ L=\frac{1}{2}\begin{pmatrix}1&-i\\-i&-1\end{pmatrix}$$

with Lie bracket relations $[R,L]=-iH$, $[H,R]=2iR$, $[H,L]=-2iL$. This is, of course, not a standard basis of $\mathfrak{sl}(2,\mathbb{C})$, however if we define

$$\hat{H}:=-iH$$

then $\{\widehat{H},L,R\}$ is a standard basis for $\mathfrak{sl}(2,\mathbb{C})$. Now, the scaling of H will not affect the weight spaces, thus

$$V \text{ is } \hat{H}-\text{semisimple} \iff V \text{ is } H-\text{semisimple}$$

and

$$V \text{ is } \hat{H} \text{ multiplicity free} \iff V \text{ is } H- \text{ multiplicity free.}$$

Moreover, as $\{\hat{H},L,R\}$ is a standard basis of $\mathfrak{sl}(2,\mathbb{C})$, the Casimir operator is given as in Definition 3.7, that is

$$\mathcal{C}=\hat{H}^2+2LR+2RL.$$

This implies that the Casimir operator for the basis $\{H,L,R\}$ is given by

$$\mathcal{C}'=-H^2+2LR+2RL.$$

Thus, with respect to the basis $\{H,L,R\}$, we say that V is quasisimple if $\mathcal{C}'$ acts as a multiple of the identity on V. Finally, the scaling of H will not affect the isomorphism classes occurring in Theorem 3.10, so the result will still hold for $\mathfrak{sl}(2,\mathbb{C})-$modules where $\mathfrak{sl}(2,\mathbb{C})$ has basis $\{H,L,R\}$. We will refer to L as the lowering operator and R as the raising operator (as they lower/raise the weight by $\pm 2i$).

4 The representation theory of $\mathfrak{so}(m)$

In this section we develop the representation theory of $\mathfrak{so}(m)$ using the theory of weights. This will give us an explicit way to describe irreducible representations of a given highest weight. In order to describe these irreducible representations we will need to introduce the so-called Schur functors. We will then give a decomposition rule for tensor products of irreducible representations of $\mathfrak{so}(m)$, and conclude the section by giving an explicit formula for the decomposition of the representation $\mathrm{Sym}^a(V)\otimes\mathrm{Sym}^b(V)$ of $\mathfrak{so}(m)$, where V is the standard representation of $\mathfrak{so}(m)$.

This section will closely follow [8, Fulton and Harris], we will give more specific references throughout where applicable.

4.1 Weight theory of $\mathfrak{so}(m)$

This subsection will follow [8, Chapter 17]. We will now introduce the weight theory of $\mathfrak{so}(m,\mathbb{C})$. This theory will give us a very efficient way to describe the irreducible representations of $\mathfrak{so}(m,\mathbb{C})$.

Let V be an m-dimensional complex vector space and let $(\cdot,\cdot) : V\times V\to\mathbb{C}$ be a non-degenerate, symmetric bilinear form on V. The orthogonal Lie algebra $\mathfrak{so}(m)$ is given as follows:

$$\mathfrak{so}(m)=\big\{A\in\mathfrak{gl}(m,\mathbb{C})\;\big|\;(Av,w)+(v,Aw)=0\;\;\forall\, v,w\in V\big\}.$$

We now wish to introduce a basis for V and express $(\cdot,\cdot)$ in terms of this basis. Unfortunately, the cases where m is odd and where m is even differ slightly and so must be considered separately. When m is even, say $m=2n$, let $\{e_1,e_2,...,e_{2n}\}$ be a basis for V, then the quadratic form $(\cdot,\cdot)$ is given by

$$(e_i,e_{n+i})=(e_{n+i},e_i)=1,\;1\leq i\leq n,\;\;(e_i,e_j)=0\;\text{ if }\;j\not\equiv i\mod(n).$$

We may express the bilinear form as

$$(x,y)=x^TMy$$

for all $x,y\in V$ where $M=\begin{pmatrix}0 & I_n\\ I_n & 0\end{pmatrix}\in M_{2n\times 2n}(\mathbb{C})$.

Remark 4.1. Had we chosen the "obvious" bilinear form, namely $(e_i,e_j)=\delta_{ij}$, then $\mathfrak{so}(m)$ would consist of skew-symmetric matrices. Our choice of $(\cdot,\cdot)$ means that we will be able to choose the set of diagonal matrices as our Cartan subalgebra, which will make the process of defining the weights and weight spaces much more straightforward.

With respect to our chosen bilinear form $\mathfrak{so}(2n)$ corresponds to the set of matrices $X\in\mathfrak{gl}(m,\mathbb{C})$ satisfying:

$$X^T\cdot M+M\cdot X=0.$$

Direct calculation shows that

$$\mathfrak{so}(2n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}(2n, \mathbb{C}) \;\middle|\; A = -D^T, B = -B^T, C = -C^T \right\}$$

where $A, B, C, D \in M_{n\times n}(\mathbb{C})$.

When m is odd, say $m = 2n+1$, we let V have basis $\{e_1, e_2, ..., e_{2n}, e_{2n+1}\}$, then the bilinear form $(\cdot,\cdot)$ satisfies the relations:

$$(e_i, e_{n+i}) = (e_{n+i}, e_i) = 1,\ 1 \leq i \leq n,\ \ (e_{2n+1}, e_{2n+1}) = 1$$

and $(e_i, e_j) = 0$ for all other pairs i, j. We may express the bilinear form as

$$(x, y) = x^T M y$$

where $M = \begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in GL(2n+1, \mathbb{C})$. Exactly as in the even case, we have that $\mathfrak{so}(2n+1)$ corresponds to the set of matrices X satisfying

$$X^T \cdot M + M \cdot X = 0.$$

Direct calculation gives us that

$$\mathfrak{so}(2n+1) = \left\{ \begin{pmatrix} A & B & E \\ C & D & F \\ G & H & J \end{pmatrix} \in \mathfrak{gl}(2n+1, \mathbb{C}) \;\middle|\; A = -D^T, B = -B^T, \right.$$
$$\left. C = -C^T, E = -H^T, F = -G^T, J = 0. \right\}$$

where $A, B, C, D \in M_{n\times n}(\mathbb{C})$. $E, F \in M_{n\times 1}(\mathbb{C})$, $G, H \in M_{1\times n}(\mathbb{C})$. $J \in M_{1\times 1}(\mathbb{C}) \cong \mathbb{C}$.

Now, in both the odd and the even cases we may take our Cartan subalgebra $\mathfrak{h}$ to be the subalgebra of diagonal matrices. Recall that we defined E_{ij} to be the matrix with a 1 in entry (i, j) and 0s elsewhere. Then $\mathfrak{h}$ is generated by the n $m \times m$ matrices

$$H_i := E_{i,i} - E_{n+i,n+i}.$$

Note that $\mathfrak{h}$ is the same in both the odd and even case. Observe that the action of H_i on V fixes e_i, sends e_{n+i} to it's negative and kills all other basis vectors. Finally, we will let $\mathfrak{h}^*$, the space of linear functionals on $\mathfrak{h}$, have basis $\{L_j\}_{j=1}^n$ such that $L_j(H_i) = \delta_{ij}$.

Definition 4.2. Let $\mathfrak{h}$ act on $\mathfrak{so}(m)$ via the adjoint action. We call $X \in \mathfrak{so}(m)$ a root vector if

$$[H, X] = \alpha(H)X$$

for all $H \in \mathfrak{h}$, where $\alpha \in \mathfrak{h}^*$. We call $\alpha \in \mathfrak{h}^*$ a root if there exists a non-zero vector $X \in \mathfrak{so}(m)$ such that $[H, X] = \alpha(H)X$ for all $H \in \mathfrak{h}$.

Lemma 4.3. *[8, Chapter 18.1, page 270.] Let $\mathfrak{h} \in \mathfrak{so}(m)$ be the Cartan subalgebra as above. Let $\mathfrak{h}$ act on $\mathfrak{so}(m)$ via the adjoint action. Then we have the following:*

1. *Let $m = 2n$ be even, then the root vectors are*
$$\{X_{i,j},\ Y_{i,j},\ Z_{i,j},\ \ 1 \leq i,j \leq n\}$$
where $X_{i,j} = E_{i,j} - E_{n+j,n+i}$, $Y_{i,j} = E_{i,n+j} - E_{j,n+i}$, $Z_{i,j} = E_{n+i,j} - E_{n+j,i}$ with roots $\{\pm L_i \pm L_j\}_{i \neq j} \subset \mathfrak{h}^*$.

2. *Let $m = 2n+1$, then all root vectors of $\mathfrak{so}(2n)$, when viewed as elements of $\mathfrak{so}(2n+1)$ via the upper left block inclusion, are still root vectors with the same roots. In addition we have root vectors*
$$\{U_i,\ \ V_i,\ \ 1 \leq i \leq n\}$$
where $U_i = E_{i,2n+1} - E_{2n+1,n+i}$, $V_i = E_{n+i,2n+1} - E_{2n+1,i}$ with roots $\{\pm L_i\} \in \mathfrak{h}^*$.

We may choose an ordering on the roots R by defining a linear functional l on $\mathfrak{h}^*$ as follows:
$$l\left(\sum_{i=1}^{n} a_i L_i\right) = c_1 a_1 + \ldots + c_n a_n$$
where $c_1 > c_2 > \ldots > c_n$. We call a root α positive if $l(\alpha) > 0$. With respect to this linear functional $\mathfrak{so}(m)$ has positive roots given as follows:

1. If $m = 2n$ then the positive roots are
$$R^+ = \{L_i + L_j\}_{i<j} \cup \{L_i - L_j\}_{i<j}.$$

2. If $m = 2n+1$ then the positive roots are
$$R^+ = \{L_i + L_j\}_{i<j} \cup \{L_i - L_j\}_{i<j} \cup \{L_i\}_i.$$

A root vector with a positive root is known as a positive root vector.

Definition 4.4. Let V be a representation of $\mathfrak{so}(m)$. We call $v \in V$ a weight vector if
$$H \cdot v = \alpha(H)v$$
for all $H \in \mathfrak{h}$, where $\alpha \in \mathfrak{h}^*$. We say $\alpha \in \mathfrak{h}^*$ is a weight of the representation if there exists a vector $v \in V$ such that $H \cdot v = \alpha(H)v$. Furthermore, if $\alpha \in \mathfrak{h}^*$ is a weight, then we define the weight space of α, denoted V_α, as
$$V_\alpha = \big\{v \in V \mid H \cdot v = \alpha(H)v\big\}.$$
Finally, we say $\alpha \in \mathfrak{h}^*$ is a highest weight if for every positive root vector $R \in \mathfrak{h}$ we have
$$H \cdot R(v) = 0.$$
for all $v \in V_\alpha$. We call v a highest weight vector.

Remark 4.5. A root (vector) is a weight (vector) in the case that $V = \mathfrak{so}(m)$ is the adjoint representation.

Let $m = 2n$ or $2n + 1$, then we have seen that $\{L_i\}_{i=1}^n$ form a basis for $\mathfrak{h}^*$. Thus we can express any weight in the form

$$\lambda_1 L_1 + \lambda_2 L_2 + ... + \lambda_n L_n \in \mathfrak{h}^*.$$

We can write this as a partition, namely $\lambda = (\lambda_1, ..., \lambda_n)$. We now state the following result:

Proposition 4.6. *[8, Theorem 19.22] Given a partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ such that $\lambda_1 \geq ... \geq \lambda_n \geq 0$, there exists an irreducible representation of $\mathfrak{so}(m)$ with highest weight $\lambda_1 L_1 + ... + \lambda_n L_n$.*

Furthermore, when $m = 2n$ we have that for every $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ such that $\lambda_1 \geq ... \geq \lambda_{n-1} \geq |\lambda_n|$, there exists an irreducible representation of $\mathfrak{so}(m)$ with highest weight $\lambda_1 L_1 + ... + \lambda_n L_n$.

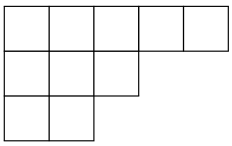
Remark 4.7. This does not give us all the irreducible representations of $\mathfrak{so}(m)$. There exist so-called spin representations which we have not mentioned. However these will not appear in the report so we have omitted any discussion about them.

4.2 Schur functors and the irreducible representations of $\mathfrak{so}(m)$

In this subsection we define Young symmetrisers and Schur functors, and give a construction of the irreducible representations of $\mathfrak{so}(m)$. We follow [8, Chapter 4] for the definition of Young symmetrisers, [8, Chapter 6] for Schur functors, and [8, Chapter 17.3] for the construction of the irreducible representations.

Definition 4.8 (Young Diagram)**.** Given a partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ such that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq 0$ we may associate to it a Young diagram of shape λ. That is, a diagram with λ_i boxes in the i^{th} row such that the rows of boxes line up on the left.

Example 4.9. Let $\lambda = (5, 3, 2)$. Then the associated Young diagram is



Given a Young diagram of a partition of d, we may number the boxes in ascending order from the top left to the bottom right. For example, for the Young diagram above, which is a partition of $5 + 3 + 2 = 10$, we number it as follows:

1	2	3	4	5
6	7	8		
9	10			

Now, given a numbered Young diagram of a partition of d, we define two subgroups of S_d, the symmetric group on d elements, denoted P_λ and Q_λ, as follows:

$$P_\lambda := \{\sigma \in S_d : \sigma \text{ preserves the numbers in each row of the diagram}\},$$

$$Q_\lambda := \{\sigma \in S_d : \sigma \text{ preserves the numbers in each column of the diagram}\}.$$

Given P_λ and Q_λ we define two elements of the group algebra $\mathbb{C}[S_d]$ as follows:

$$a_\lambda := \sum_{\sigma \in P_\lambda} e_\sigma \quad \text{and} \quad b_\lambda := \sum_{\sigma \in Q_\lambda} \operatorname{sgn}(\sigma) e_\sigma.$$

We now define the Young symmetriser of a partition.

Definition 4.10 (Young symmetriser)**.** For a partition λ, let a_λ and b_λ be defined as above. Then

$$c_\lambda = a_\lambda \cdot b_\lambda \in \mathbb{C}[S_d]$$

is the Young symmetriser of λ.

Example 4.11. We now give a full example of the construction of the Young symmetriser for a given partition λ. For large d the Young symmetriser will contain a vast number of terms. With this in mind let $\lambda = (2, 2, 1)$ be a partition of $d = 5$. Then λ has associated numbered Young diagram:

1	2
3	4
5	

Now, P_λ, Q_λ, a_λ and b_λ are given as follows:

$$\begin{aligned}
P_\lambda &= \big\{(1), (12), (34)\big\},\\
Q_\lambda &= \big\{(1), (13), (15), (35), (135), (153), (24)\big\},\\
a_\lambda &= \sum_{\sigma \in P_\lambda} e_\sigma = e_{(1)} + e_{(12)} + e_{(34)},\\
b_\lambda &= \sum_{\sigma \in Q_\lambda} \operatorname{sgn}(\sigma) e_\sigma = e_{(1)} - e_{(13)} - e_{(15)} - e_{(35)} - e_{(24)} + e_{(135)} + e_{(153)}.
\end{aligned}$$

Thus, we have that the Young symmetriser of $\lambda = (2, 2, 1)$ is

$$\begin{aligned}
c_\lambda = a_\lambda \cdot b_\lambda &= \sum_{\sigma \in P_\lambda} e_\sigma \cdot \sum_{\tau \in Q_\lambda} \operatorname{sgn}(\tau) e_\tau\\
&= e_{(1)} - e_{(13)} - e_{(15)} - e_{(35)} - e_{(24)} + e_{(135)} + e_{(153)}\\
&+ e_{(12)} - e_{(132)} - e_{(152)} - e_{(12)(35)} - e_{(124)} + e_{(1352)} + e_{(1532)}\\
&+ e_{(34)} - e_{(143)} - e_{(34)(15)} - e_{(354)} - e_{(234)} + e_{(1435)} + e_{(1543)}.
\end{aligned}$$

Now, having defined the Young symmetriser we turn to the definition of the Schur functor, we follow [8, Chapter 6]. Let V be a complex vector space, then V is a representation of $GL(V)$ by definition. Now consider $V^{\otimes d} := \underbrace{V \otimes V \otimes ... \otimes V}_{\text{d times}}$, then S_d acts on $V^{\otimes d}$ on the right[2] by permuting factors:

$$(v_1 \otimes v_2 \otimes ... \otimes v_d) \cdot \sigma = v_{\sigma(1)} \otimes \sigma_{\sigma(2)} \otimes ... \otimes v_{\sigma(d)}$$

for all $\sigma \in S_d$, $v_1 \otimes v_2 \otimes ... \otimes v_d \in V^{\otimes d}$. Note that the action of S_d commutes with the left action of $GL(V)$. We now give the following definition:

Definition 4.12 (Schur functor)**.** Let λ be a partition of d and let c_λ be the Young symmetriser of λ. Then c_λ acts on $V^{\otimes d}$ on the right as above and we define

$$\mathbb{S}_\lambda(V) = \operatorname{im}\left(c_\lambda : V^{\otimes d} \to V^{\otimes d}\right).$$

$\mathbb{S}_\lambda(V)$ is known as a Schur functor and is also a representation of $GL(V)$, since the action of S_d commutes with the action of $GL(V)$ on $V^{\otimes d}$.

Example 4.13. Let $V = \mathbb{C}^n$ with basis $\{e_1, e_2, ..., e_n\}$. Fix $d \in \mathbb{Z}_{\geq 0}$.

1. Let $\lambda = (d)$. Then
$$c_\lambda = c_{(d)} = a_{(d)} = \sum_{\sigma \in S_d} e_\sigma$$
and so $\mathbb{S}_{(d)}(V)$ is the subspace of $V^{\otimes d}$ given by elements of the form
$$\sum_{\sigma \in S_d} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes ... \otimes v_{\sigma(d)}.$$
Thus, $\mathbb{S}_{(d)}(V) = \operatorname{Sym}^d(V)$.

2. Let $\lambda = (\underbrace{1, 1, ..., 1}_{\text{d times}})$. Then
$$c_\lambda = c_{(1,1,..,1)} = b_{(1,1,...1)} = \sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) e_\sigma$$
and $\mathbb{S}_{(1,1,...,1)}(V)$ is the subspace of $V^{\otimes d}$ given by elements of the form
$$\sum_{\sigma \in S_d} \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes ... \otimes v_{\sigma(d)}.$$
Thus $\mathbb{S}_{(1,1,...,1)}(V) = \bigwedge^d(V)$.

[2] We act on the right as acting with the analogous left action involves taking inverses - something we happily avoid.

We now turn to the last objective of this subsection, namely describing the irreducible representations of $\mathfrak{so}(m)$ which will be of interest to us in this report. This description is known as "Weyl's construction for Orthogonal groups" and we follow [8, Chapter 19.5].

For the remainder of this subsection we let $V = \mathbb{C}^m$. Given a bilinear form $(\cdot,\cdot)$ and a pair (i,j) with $1 \leq i < j \leq d$ we may define a contraction from $V^{\otimes d}$ to $V^{\otimes(d-2)}$ as follows:

$$\begin{aligned} \Psi_{i,j} : V^{\otimes d} &\longrightarrow V^{\otimes(d-2)} \\ v_1 \otimes ... \otimes v_d &\longmapsto (v_i, v_j) v_1 \otimes ... \widehat{v_i} \otimes ... \otimes \widehat{v_j} \otimes ... \otimes v_d \end{aligned}$$

where $\widehat{v_i}$ denotes the omission of the entry v_i. We also define

$$V^{[d]} := \bigcap_{1\leq i<j\leq d} \ker\left(\Psi_{i,j}\right),$$

that is, $V^{[d]}$ is the intersection of the kernels of all possible contractions from $V^{\otimes d}$ to $V^{\otimes(d-2)}$. Now, for any partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ of d such that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq 0$ let

$$\mathbb{S}_{[\lambda]}(V) := V^{[d]} \cap \mathbb{S}_\lambda(V).$$

Now, one sees that the contractions $\Psi_{i,j}$ are $\mathfrak{so}(m,\mathbb{C})$-homomorphisms, therefore $V^{[d]}$ is a representation of $\mathfrak{so}(m,\mathbb{C})$, as is $\mathbb{S}_\lambda(V)$, thus $\mathbb{S}_{[\lambda]}(V)$ is a representation of $\mathfrak{so}(m,\mathbb{C})$. We conclude this subsection with the following three results:

Proposition 4.14. *[8, Exercise 19.21]*

1. *The kernel of the contraction from* $\mathrm{Sym}^d(V)$ *to* $\mathrm{Sym}^{d-2}(V)$ *is the representation* $\mathbb{S}_{[d]}(V)$ *of* $\mathfrak{so}(m,\mathbb{C})$ *with highest weight* dL_1.

2. *We have the decomposition*

$$\mathrm{Sym}^d(V) = \mathbb{S}_{[d]}(V) \oplus \mathbb{S}_{[d-2]}(V) \oplus ... \oplus \mathbb{S}_{[d-2p]}(V)$$

where p is the largest integer such that $p \leq \frac{d}{2}$.

Proof. As this result is given as an exercise in Fulton & Harris we provide a proof.

1. Let $v_1v_2...v_d$ denote the element $\sum_{\sigma\in S_d} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes ... \otimes v_{\sigma(d)} \in \mathrm{Sym}^d(V)$. Then the contraction, Ψ, from $\mathrm{Sym}^d(V)$ to $\mathrm{Sym}^{d-2}(V)$ is given by

$$\Psi(v_1v_2...v_d) = (v_1, v_2)v_3...v_d$$

or equivalently

$$\Psi\left(\sum_{\sigma\in S_d} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes ... \otimes v_{\sigma(d)}\right) = \sum_{\sigma\in S_d} \big(v_{\sigma(1)}, v_{\sigma(2)}\big) v_{\sigma(3)} \otimes ... \otimes v_{\sigma(d)}.$$

We know that $\mathbb{S}_{[d]}(V) = V^{[d]} \cap \mathbb{S}_d(V) = V^{[d]} \cap \mathrm{Sym}^d(V)$. Let $K \subset \mathrm{Sym}^d(V)$ denote the kernel of the contraction Ψ. We will show that $K = V^{[d]} \cap \mathrm{Sym}^d(V)$.

Firstly, we show $V^{[d]} \cap \mathrm{Sym}^d(V) \subset K$. Indeed, if $v_1 v_2 ... v_d \in V^{[d]}$ then for any pair (i,j) such that $1 \leq i < j \leq d$ we have by linearity

$$\sum_{\sigma \in S_d} \big(v_{\sigma(i)}, v_{\sigma(j)}\big)\, v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes ... \otimes \widehat{v_{\sigma(i)}} \otimes ... \otimes \widehat{v_{\sigma(j)}} ... \otimes v_{\sigma(d)} = 0.$$

In particular for $(i,j) = (1,2)$ we have

$$\sum_{\sigma \in S_d} \big(v_{\sigma(1)}, v_{\sigma(2)}\big)\, v_{\sigma(3)} ... \otimes v_{\sigma(d)} = 0,$$

therefore $v_1 v_2 ... v_d \in K$.

Conversely, suppose $v_1 v_2 ... v_d = \sum_{\sigma \in S_d} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes ... \otimes v_{\sigma(d)} \in K$, then

$$\sum_{\sigma \in S_d} \big(v_{\sigma(1)}, v_{\sigma(2)}\big) v_{\sigma(3)} \otimes ... \otimes v_{\sigma(d)} = 0$$

so certainly $v_1 v_2 ... v_d \in \ker(\Psi_{1,2})$. Now, given (i,j), set $\tau = \big(\sigma(1), \sigma(i)\big)\big(\sigma(2), \sigma(j)\big)\sigma$ then we have

$$\begin{aligned}
&\sum_{\sigma \in S_d} \big(v_{\sigma(i)}, v_{\sigma(j)}\big)\, v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes ... \otimes \widehat{v_{\sigma(i)}} \otimes ... \otimes \widehat{v_{\sigma(j)}} ... \otimes v_{\sigma(d)} \\
&= \sum_{\tau \in S_d} \big(v_{\tau(1)}, v_{\tau(2)}\big)\, v_{\tau(i)} \otimes v_{\tau(j)} \otimes v_{\tau(3)} \otimes ... \otimes v_{\tau(d)} \\
&= \sum_{\tau' \in S_d} \big(v_{\tau'(1)}, v_{\tau'(2)}\big)\, v_{\tau'(3)} \otimes ... \otimes v_{\tau'(d)} \\
&= \sum_{\sigma \in S_d} \big(v_{\sigma(1)}, v_{\sigma(2)}\big)\, v_{\sigma(3)} \otimes ... \otimes v_{\sigma(d)} = 0
\end{aligned}$$

where $\tau' = r\tau$ where $r \in S_d$ is such that

$$\big(\tau(i), \tau(j), \tau(3), ..., \tau(d)\big) \overset{r}{\mapsto} \big(\tau(3), \tau(4), ..., \tau(d)\big).$$

Thus $v_1 v_2 ... v_d \in \ker(\Psi_{i,j})$ for all (i,j) and so $K \subset V^{[d]} \cap \mathrm{Sym}^d(V)$.

We conclude by observing that $e_1 e_1 ... e_1 \in K$ has weight dL_1, thus $K = \mathbb{S}_{[d]}(V)$ is indeed the representation with highest weight dL_1, as required.

2. This follows from induction and part 1, which tells us that since the contraction is surjective we have $\mathrm{Sym}^d(V) = \mathbb{S}_{[d]}(V) \oplus \mathrm{Sym}^{d-2}(V)$ as representations.

$\square$

Proposition 4.15. *[8, Theorem 19.22]*

1. *If* $m = 2n+1$ *and* $\lambda = (\lambda_1, ..., \lambda_n)$, $\lambda_1 \geq ... \geq \lambda_n \geq 0$, *then* $\mathbb{S}_{[\lambda]}(V)$ *is the irreducible representation with highest weight* $\lambda_1 L_1 + ... + \lambda_n L_n$.

2. *If* $m = 2n$ *and* $\lambda = (\lambda_1, ..., \lambda_n)$, $\lambda_1 \geq ... \geq \lambda_{n-1} \geq 0$, $\lambda_n = 0$ *then* $\mathbb{S}_{[\lambda]}(V)$ *is the irreducible representation with highest weight* $\lambda_1 L_1 + ... + \lambda_{n-1} L_{n-1}$.

3. *If* $m = 2n$ *and* $\lambda = (\lambda_1, ..., \lambda_n)$, $\lambda_1 \geq ... \geq \lambda_n > 0$, *then* $\mathbb{S}_{[\lambda]}(V)$ *decomposes into the direct sum of two irreducible representations with highest weights* $\lambda_1 L_1 + ... + \lambda_n L_n$ *and* $\lambda_1 L_1 + ... + \lambda_{n-1} L_{n-1} - \lambda_n L_n$.

Proposition 4.16. *[8, Theorem 19.2, Theorem 19.14]*

1. *Let* $m = 2n + 1$ *be odd. Then for* $k = 1, ..., n$, *the exterior power* $\bigwedge^k(V)$ *of the standard representation* V *of* $\mathfrak{so}(m, \mathbb{C})$ *is the irreducible representation with highest weight* $L_1 + ... + L_k$.

2. *Let* $m = 2n$ *be even then:*

   (a) *If* $k = 1, 2, ..., n - 1$ *then the exterior power* $\bigwedge^k(V)$ *of the standard representation* V *of* $\mathfrak{so}(m, \mathbb{C})$ *is the irreducible representation with highest weight* $L_1 + ... + L_k$.

   (b) *If* $k = n$, *then* $\bigwedge^k(V)$ *has exactly two irreducible factors. (Note: Proposition 4.15 exactly describes these irreducible factors.)*

4.3 A decomposition rule for tensor products of irreducible representations

We have now developed a way to describe the irreducible representations of $\mathfrak{so}(m)$ which are of interest to us. We now look towards the final part of the section, namely, looking at a decomposition rule for tensor products of irreducible representations.

Definition 4.17 (Littlewood-Richardson coefficients)**.** [8, Appendix A, Page 456.] Let λ, μ and ν be partitions. We define the Littlewood-Richardson coefficient $M^{\nu}_{\lambda,\mu}$ to be the number of ways to expand the Young diagram of λ to the Young diagram of ν using a strict $\mu-$expansion.

That is, if $\mu = (\mu_1, ..., \mu_k)$ then a $\mu-$expansion of a Young diagram is obtained by first adding μ_1 boxes to the diagram such that no two boxes are in the same column and then putting a number 1 in them, then adding μ_2 boxes to the diagram such that no two of these boxes are in the same column and putting a number 2 in them and so on.

The expansion is called strict if when the integers in the boxes are listed from right to left starting on the top row and working down and one considers the first t entries in the list for any $1 \leq t \leq \mu_1 + ... + \mu_k$, then the number 1 occurs at least as many times as the number 2 occurs, and the number 2 occurs at least as many times as the number 3 occurs, and so on.

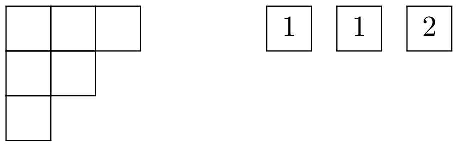
Example 4.18. We now see some examples of strict expansions of Young diagrams and computing Littlewood-Richardson coefficients.

1. Let $\lambda = (3,2,1)$, $\mu = (2,1)$ and $\nu = (4,2,2)$. Then $M^{\nu}_{\lambda\mu} = 0$. Indeed, if we strictly $\mu-$expand the Young diagram of shape λ then we will have

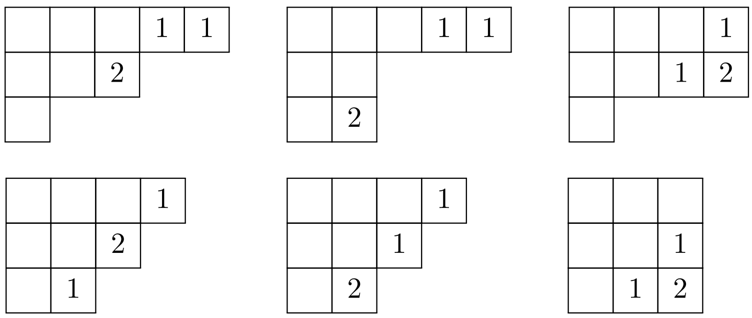
$$\sum_i \lambda_i + \sum_j \mu_j = (3+2+1) + (2+1) = 9$$

   total boxes, however the Young diagram for ν only has $\nu_1 + \nu_2 + \nu_3 = 4+2+2 = 8$ boxes. Thus there is no way to $\mu-$expand the Young diagram of shape λ to the Young diagram of ν.

2. Let $\lambda = (3,2,1)$, $\mu = (2,1)$ and $\nu = (5,2,2)$. Then $M^{\nu}_{\lambda\mu} = 1$. Indeed we begin with a Young diagram for λ and wish to add 2 boxes with a 1 in them, and 1 box with a 2 in it according to the above criteria.



   The Young diagrams we can end up with are as follows



   As we can see the Young diagram for $\nu = (5,2,2)$ appears once, so indeed $M^{\nu}_{\lambda\mu} = 1$.

We are now ready to introduce the decomposition rule for tensor products of $\mathfrak{so}(m)$ representations. Using this we will develop a decomposition rule for representations of the form $\mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(V)$. For the remainder of this section we let Γ_λ denote the irreducible representation with highest weight λ. That is, $\Gamma_\lambda = \mathbb{S}_{[\lambda]}(V)$, unless $m = 2n$ and $\lambda_n > 0$, in which case $\mathbb{S}_{[\lambda]}(V) = \Gamma_\lambda \oplus \Gamma_{(\lambda_1,\ldots,\lambda_{n-1},-\lambda_n)}$ (see Proposition 4.15).

Lemma 4.19. *[8, Section 25.3] Let λ and μ be partitions such that $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$ and $\mu_1 \geq \ldots \geq \mu_n \geq 0$. Then we have*

$$\Gamma_\lambda \otimes \Gamma_\mu = \bigoplus_\nu N^{\nu}_{\lambda\mu}\Gamma_\nu$$

where

$$N^{\nu}_{\lambda\mu} = \sum_{\zeta,\sigma,\tau} M^{\lambda}_{\zeta\sigma}\, M^{\mu}_{\zeta\tau}\, M^{\nu}_{\sigma\tau}$$

where the M's are Littlewood-Richardson coefficients, and where we sum over all possible partitions ζ, σ, τ.

We now use this result to give an explicit decomposition for representations of the form $\mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(V)$. We begin with the following definition.

Definition 4.20. Let $Y(m, n, \zeta)$ be the set of partitions of $m + n - 2\zeta$ such that the first entry is greater than to equal to $m - \zeta$, the second entry is less than or equal to $n - \zeta$ and all other entries are 0.

Example 4.21. $Y(4, 3, 1)$ = "the set of partitions of 5, such that the first entry is ≥ 3, the second entry is ≤ 2, and all other entries are 0" $= \{(5), (4, 1), (3, 2)\}$.

Theorem 4.22. *Let $m \geq 4$ and V be the standard representation of $\mathfrak{so}(m)$. Then we have the following decomposition:*

$$\mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(V) = \bigoplus_{l=0}^{l \leq \frac{a}{2}} \left(\bigoplus_{k \geq \frac{b-(a-2l)}{2}}^{k \leq \frac{b}{2}} \bigoplus_{0 \leq \zeta \leq b-2k} \bigoplus_{\alpha \in Y(a-2l, b-2k, \zeta)} \Gamma_\alpha \right.$$
$$\left. \bigoplus \bigoplus_{k=0}^{k < \frac{b-(a-2l)}{2}} \bigoplus_{0 \leq \zeta \leq a-2l} \bigoplus_{\alpha \in Y(b-2k, a-2l, \zeta)} \Gamma_\alpha \right).$$

Proof. We begin by finding the decomposition of $\Gamma_\lambda \otimes \Gamma_\mu$ where $\lambda = (p, 0, ..., 0)$ and $\mu = (q, 0, .., 0)$. Note that $\Gamma_\lambda = \mathbb{S}_{[p]}(V)$ and $\Gamma_\mu = \mathbb{S}_{[q]}(V)$. We assume w.l.o.g. that $p \geq q$. We have

$$\Gamma_\lambda \otimes \Gamma_\mu = \bigoplus_{\nu} N^{\nu}_{\lambda\mu} \Gamma_\nu, \qquad N^{\nu}_{\lambda\mu} = \sum_{\zeta,\sigma,\tau} M^{\lambda}_{\zeta\sigma}\, M^{\mu}_{\zeta\tau}\, M^{\nu}_{\sigma\tau}.$$

We now look for all ζ, σ, τ, ν such that $N^{\nu}_{\lambda,\mu}$ is non-zero.

- For $M^{\lambda}_{\zeta\sigma} \neq 0$ we require $\zeta = (\zeta, 0, , , 0)$ [3], $\zeta \leq p$ and $\sigma = (p - \zeta, 0, ..., 0)$, since $\lambda = (p, 0, ..., 0)$.
- For $M^{\mu}_{\zeta\tau} \neq 0$ we require $\zeta = (\zeta, 0, , , 0)$, $\zeta \leq q$ and $\tau = (q - \zeta, 0, ..., 0)$, since $\mu = (q, 0, ..., 0)$.
- As we assumed $q \leq p$, we require $\zeta \leq q$. Note that σ and τ are fixed once ζ is chosen. Suppose now we fix $0 \leq \zeta \leq q$.

[3] Abuse of notation: We will denote the partition and the first and potentially only non-zero value of the partition with the same symbol.

- $M^{\nu}_{\sigma\tau}$ is the number of ways to extend the Young diagram for $\sigma = (p-\zeta, 0, ..., 0)$ via a strict τ-expansion to the Young diagram for ν, where $\tau = (q-\zeta, 0..., 0)$.

$$\underbrace{\boxed{}\boxed{\cdots}\boxed{\cdots}\boxed{\cdots}\boxed{}}_{p-\zeta \text{ boxes}} \qquad \underbrace{\boxed{1}\;\boxed{1}\;\cdots\;\boxed{1}}_{q-\zeta \text{ boxes with a 1 in them}}$$

  Thus, the partitions ν which can be achieved are exactly the partitions in $Y(p, q, \zeta)$, that is, the set of partitions of $p+q-2\zeta$ such that the first entry is greater than to equal to $p-\zeta$, the second entry is less than or equal to $q-\zeta$ and all other entries are 0. Note that each of these partitions appears exactly once.

- Altogether this gives us the decomposition

$$\Gamma_\lambda \otimes \Gamma_\mu = \mathbb{S}_{[p]}(V) \otimes \mathbb{S}_{[q]}(V) = \bigoplus_{0\leq\zeta\leq q} \bigoplus_{\nu\in Y(p,q,\zeta)} \Gamma_\nu$$

Now, we recall that we have $\mathrm{Sym}^d(V) = \bigoplus_{l=0}^{l\leq\frac{d}{2}} \mathbb{S}_{[d-2l]}(V)$. Thus,

$$\begin{aligned}\mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(V) &= \left(\bigoplus_{l=0}^{l\leq\frac{a}{2}} \mathbb{S}_{[a-2l]}(V)\right) \bigotimes \left(\bigoplus_{k=0}^{k\leq\frac{b}{2}} \mathbb{S}_{[b-2k]}(V)\right) \\ &= \bigoplus_{l=0}^{l\leq\frac{a}{2}} \bigoplus_{k=0}^{k\leq\frac{b}{2}} \mathbb{S}_{[a-2l]}(V) \otimes \mathbb{S}_{[b-2k]}(V).\end{aligned}$$

We wish to substitute in our formula for the decomposition of $\mathbb{S}_{[p]}(V) \otimes \mathbb{S}_{[q]}(V)$ found above, but recall that we assumed $p \geq q$, therefore we must split the above double summation into two double summations, one where $a-2l \geq b-2k \iff k \geq \frac{b-(a-2l)}{2}$ and the other where $a-2l < b-2k \iff k < \frac{b-(a-2l)}{2}$. This gives:

$$\begin{aligned}\mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(V) = \bigoplus_{l=0}^{l\leq\frac{a}{2}} \Bigg(&\bigoplus_{k\geq\frac{b-(a-2l)}{2}}^{k\leq\frac{b}{2}} \mathbb{S}_{[a-2l]}(V) \otimes \mathbb{S}_{[b-2k]}(V) \\ &\bigoplus \bigoplus_{k=0}^{k<\frac{b-(a-2l)}{2}} \mathbb{S}_{[b-2k]}(V) \otimes \mathbb{S}_{[a-2l]}(V)\Bigg).\end{aligned}$$

Finally, substituting in the decompositions of $\mathbb{S}_{[a-2l]}(V) \otimes \mathbb{S}_{[b-2k]}(V)$ and $\mathbb{S}_{[b-2k]}(V) \otimes \mathbb{S}_{[a-2l]}(V)$ gives the result.

$\square$

5 Computing the relative Lie algebra cohomology of the Weil representation of $\mathfrak{o}(p,1) \times \mathfrak{sl}(2)$ with coefficients in $\mathrm{Sym}^l(V)$

Let $G = SO_0(p,1)$; $K = SO(p) \times SO(1) \cong SO(p)$, the maximal compact subgroup of $SO_0(p,1)$. Then $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{o}(p,1)$, $\mathfrak{k} = \mathrm{Lie}(K) = \mathfrak{so}(p) \oplus \mathfrak{so}(1) \cong \mathfrak{so}(p)$. Recall that we have the splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_0$. Let $\mathfrak{p} = \mathfrak{p}_0 \otimes \mathbb{C}$ be the complexification of $\mathfrak{p}_0$. Furthermore, let $\mathcal{F} = \mathbb{C}[z_1, ..., z_p, z_{p+1}]$ denote the Fock model of the Weil representation; and let V denote the standard representation of G. Recall from Section 2 that we have the isomorphism:

$$C^i\Big(\mathfrak{g}, K; \mathcal{F} \otimes \mathrm{Sym}^l(V)\Big) \cong C^i\Big(\mathfrak{o}(p,1,\mathbb{C}), K_{\mathbb{C}}; \mathcal{F} \otimes \mathrm{Sym}^l(V \otimes \mathbb{C})\Big).$$

In this section we will compute the spaces

$$C_l^i := C^i\big(\mathfrak{o}(p,1,\mathbb{C}), K_{\mathbb{C}}; \mathcal{F} \otimes \mathrm{Sym}^l(V \otimes \mathbb{C})\big) = \Big[\mathcal{F} \otimes \bigwedge^i \mathfrak{p}^* \otimes \mathrm{Sym}^l(V \otimes \mathbb{C})\Big]^{SO(p \otimes \mathbb{C})}$$

in the cases $l = 0, 1$ and subsequently compute the cohomology groups.

Remark 5.1. For ease of notation, throughout the section we will drop the "$\otimes\mathbb{C}$" with the understanding that it is there. For example, when reading $\Big[\mathcal{F} \otimes \bigwedge^i \mathfrak{p}^* \otimes V\Big]^{SO(p)}$, one should interpret it as $\Big[\mathcal{F} \otimes \bigwedge^i \mathfrak{p}^* \otimes (V \otimes \mathbb{C})\Big]^{SO(p \otimes \mathbb{C})}$.

We begin by discussing two $\mathfrak{k}$-isomorphisms which will be used extensively throughout the section.

Lemma 5.2. *We have the following isomorphism of $\mathfrak{k}$ representations:*

$$\mathfrak{p}_0 \cong V.$$

Remark 5.3. This implies that as $\mathfrak{k}_{\mathbb{C}}$ representations we have $\mathfrak{p} \cong V \otimes \mathbb{C}$.

Proof. Recall that $\mathfrak{p}_0$ has basis $\{X_{\gamma,p+1}\}$ and that $\mathfrak{k}$ acts on $\mathfrak{p}_0$ via the adjoint representation, 'ad'. We have:

$$[X_{\alpha,\beta}, X_{\gamma,p+1}] = -\delta_{\beta,\gamma}(E_{\alpha,p+1} + E_{p+1,\alpha}) + \delta_{\alpha,\gamma}(E_{\beta,p+1} + E_{p+1,\beta})$$

Thus

$$\mathrm{ad}_{X_{\alpha,\beta}}(X_{\gamma,p+1}) = [X_{\alpha,\beta}, X_{\gamma,p+1}] = \begin{cases} X_{\beta,p+1} & \text{if } \gamma = \alpha \\ -X_{\alpha,p+1} & \text{if } \gamma = \beta \\ 0 & \text{otherwise.} \end{cases}$$

As V is the standard representation of $\mathfrak{k}$ we have:

$$X_{\alpha,\beta} \cdot e_\gamma = \begin{cases} e_\beta & \text{if } \gamma = \alpha \\ -e_\alpha & \text{if } \gamma = \beta \\ 0 & \text{otherwise.} \end{cases}$$

Let T be the linear map

$$T : \mathfrak{p}_0 \to V$$
$$X_{\alpha,p+1} \mapsto e_\alpha.$$

It is clear that T is an isomorphism of vector spaces. We also see that

$$T\left(\left[X_{\alpha,\beta}, X_{\gamma,p+1}\right]\right) = X_{\alpha,\beta} \cdot T\left(X_{\gamma_p+1}\right)$$

for all $X_{\gamma,p+1} \in \mathfrak{p}_0$, $X_{\alpha,\beta} \in \mathfrak{k}$ and so T is a $\mathfrak{k}$-isomorphism. □

Definition 5.4 (Hodge star)**.** Let $\mathfrak{p}^*$ have canonical ordered basis $\{\omega_\alpha\}_{\alpha=1}^{p}$. We define the following map from $\bigwedge^i \mathfrak{p}^*$ to $\bigwedge^{p-i} \mathfrak{p}^*$:

$$\begin{aligned} \star : \bigwedge^i \mathfrak{p}^* &\to \bigwedge^{p-i} \mathfrak{p}^* \\ \omega &\mapsto \star(\omega) \end{aligned}$$

such that $\omega \wedge \star(\omega) = \omega_1 \wedge \omega_2 \wedge ... \wedge \omega_p \in \bigwedge^p \mathfrak{p}^* \cong \mathbb{C}$.

We ultimately wish to show that $\star$ is a $\mathfrak{k}-$isomorphism, but first we show the following:

Lemma 5.5. *Let $\omega = \omega_{k_1} \wedge \omega_{k_2} \wedge ... \wedge \omega_{k_i} \in \bigwedge^i \mathfrak{p}^*$ and suppose w.l.o.g. that $k_1 < k_2 < ... < k_i$ then*

$$\star(\omega) = (-1)^\varepsilon \omega_1 \wedge ... \wedge \widehat{\omega_{k_1}} \wedge ... \wedge \widehat{\omega_{k_i}} \wedge ... \wedge \omega_p \in \bigwedge^{p-i} \mathfrak{p}^*$$

where $\widehat{\omega_{k_i}}$ denotes the omission of entry ω_{k_i} and

$$\varepsilon = \frac{(i-3)i}{2} + \sum_{j=1}^{i} k_j.$$

Proof. Consider

$$(\omega_{k_1} \wedge \omega_{k_2} \wedge ... \wedge \omega_{k_i}) \wedge (\omega_1 \wedge ... \wedge \widehat{\omega_{k_1}} \wedge ... \wedge \widehat{\omega_{k_i}} \wedge ... \wedge \omega_p)$$

We wish to swap the entries so that the indices are in ascending order when reading from left to right. We recall that by definition of the exterior power, we introduce a '-1' every time we swap two adjacent entries. We proceed by swapping term by term:

For ω_{k_1} : We shift it $(i-1) + (k_1 - 1)$ places to the right.
For ω_{k_2} : We shift it $(i-2) + (k_2 - 1)$ places to the right.
For ω_{k_3} : We shift it $(i-3) + (k_3 - 1)$ places to the right.
$\vdots$
For ω_{k_i} : We shift it $0 + (k_i - 1)$ places to the right.

Now we let

$$\varepsilon = \sum_{j=1}^{i}(i-j)+(k_j-1) = \sum_{j=0}^{i-1} j + \sum_{j=1}^{i}(k_j-1) = \frac{(i-1)i}{2} - i + \sum_{j=1}^{i} k_j = \frac{(i-3)i}{2} + \sum_{j=1}^{i} k_j.$$

Then, we have

$$(\omega_{k_1}\wedge\omega_{k_2}\wedge ... \wedge\omega_{k_i})\wedge(\omega_1\wedge ... \wedge\widehat{\omega_{k_1}}\wedge ... \wedge\widehat{\omega_{k_i}}\wedge ... \wedge\omega_p) = (-1)^{\varepsilon}\omega_1\wedge\omega_2\wedge ... \wedge\omega_{p-1}\wedge\omega_p$$
$$\implies \star(\omega) = (-1)^{\varepsilon}\,\omega_1\wedge ... \wedge\widehat{\omega_{k_1}}\wedge ... \wedge\widehat{\omega_{k_i}}\wedge ... \wedge\omega_p. \qquad \square$$

Having now obtained an explicit formula for $\star$, we state and prove the following:

Lemma 5.6.

$$\star : \bigwedge^{i}\mathfrak{p}^* \to \bigwedge^{p-i}\mathfrak{p}^*$$

is a $\mathfrak{k}\cong\mathfrak{so}(p)$ isomorphism.

Proof. It is clear that $\star$ is an isomorphism of vector spaces as both vector spaces have the same dimension and $\star$ is easily seen to be injective. To see that $\star$ is a $\mathfrak{k}-$isomorphism, first recall that $\mathfrak{k}$ acts on $\mathfrak{p}$ via the adjoint action and this induces an action of $\mathfrak{k}$ on $\mathfrak{p}^*$, which is explicitly given by

$$X_{\alpha,\beta}\cdot\omega_\gamma = \mathrm{ad}^*_{X_{\alpha,\beta}}(\omega_\gamma) = \begin{cases} \omega_\beta, \text{ if } \gamma=\alpha \\ -\omega_\alpha, \text{ if } \gamma=\beta \\ 0 \text{ , otherwise.}\end{cases}$$

Now, fix $X_{\alpha,\beta}\in\mathfrak{k}$ and let $\omega=\omega_{k_1}\wedge\omega_{k_2}\wedge ... \wedge\omega_{k_i}$. We consider the following 4 cases:

1. $\alpha\in\{k_1,k_2,...,k_i\}$ and $\beta\in\{k_1,k_2,...,k_i\}$.
   In this case we have $X_{\alpha,\beta}\cdot\omega=0$, and so $\star(X_{\alpha,\beta}\cdot\omega)=0$. On the other hand we have
   $$X_{\alpha,\beta}\cdot\big(\star(\omega)\big) = (-1)^{\varepsilon}X_{\alpha,\beta}\cdot\underbrace{(\omega_1\wedge ... \wedge\widehat{\omega_{k_1}}\wedge ... \wedge\widehat{\omega_{k_i}}\wedge ... \wedge\omega_p)}_{\omega_\alpha \text{ and } \omega_\beta \text{ appear with a hat.}} = 0$$
   Thus, $\star\cdot X_{\alpha,\beta} = X_{\alpha,\beta}\cdot\star$ as operators.
2. The proof is similar if both $\alpha,\beta\notin\{k_1,k_2,...,k_i\}$.
3. Suppose $\alpha\in\{k_1,k_2,...,k_i\}$, and $\beta\notin\{k_1,k_2,...,k_i\}$ then
   $$\star\big(X_{\alpha,\beta}\cdot(\underbrace{\omega_{k_1}\wedge ... \wedge\omega_\alpha\wedge ... \wedge\omega_{k_i}}_{\text{terms in order}})\big) = \star\big(\underbrace{\omega_{k_1}\wedge ... \wedge\omega_\beta\wedge ... \wedge\omega_{k_i}}_{\omega_\beta \text{ term out of order}}\big)$$
   $$= \star\big((-1)^{\gamma}\underbrace{\omega_{k_1}\wedge ... \wedge\omega_\beta\wedge ... \wedge\omega_{k_i}}_{\text{terms in order}}\big) \text{ where } \gamma = \#\{k_i \mid k_i \text{ lies between } \alpha \text{ and } \beta\}$$
   $$= (-1)^{\gamma}\star\big(\omega_{k_1}\wedge ... \wedge\omega_\beta\wedge ... \wedge\omega_{k_i}\big)$$
   $$= (-1)^{\gamma+\varepsilon'}\omega_1\wedge ... \wedge\widehat{\omega_{k_1}}\wedge ... \wedge\widehat{\omega_\beta}\wedge ... \wedge\widehat{\omega_{k_i}}\wedge ... \wedge\omega_p$$
   with $\varepsilon' = \dfrac{(i-3)i}{2} + \sum_{j=1}^{i} k_j - \alpha + \beta$, (as we applied $\star$ to $\omega_{k_1}\wedge ... \wedge\omega_\beta\wedge ... \wedge\omega_{k_i}$).

Now, consider $X_{\alpha,\beta} \cdot \star(\omega_{k_1} \wedge ... \wedge \omega_\alpha \wedge ... \wedge \omega_{k_i})$. We have:

$$\begin{aligned}
&X_{\alpha,\beta} \cdot \star(\omega_{k_1} \wedge ... \wedge \omega_\alpha \wedge ... \wedge \omega_{k_i}) \\
=&X_{\alpha,\beta} \cdot \big((-1)^\varepsilon \omega_1 \wedge ... \wedge \omega_\beta \wedge ... \wedge \widehat{\omega_{k_1}} \wedge ... \wedge \widehat{\omega_\alpha} \wedge ... \wedge \widehat{\omega_{k_i}} \wedge ... \wedge \omega_p\big) \\
&\text{where } \varepsilon = \frac{(i-3)i}{2} + \sum_{j=1}^{i} k_j \\
=&(-1)^\varepsilon X_{\alpha,\beta} \cdot \big(\omega_1 \wedge ... \wedge \omega_\beta \wedge ... \wedge \widehat{\omega_{k_1}} \wedge ... \wedge \widehat{\omega_\alpha} \wedge ... \wedge \widehat{\omega_{k_i}} \wedge ... \wedge \omega_p\big) \\
=&(-1)^\varepsilon\, \omega_1 \wedge ... \wedge \omega_\alpha \wedge ... \wedge \widehat{\omega_{k_1}} \wedge ... \wedge \widehat{\omega_\alpha} \wedge ... \wedge \widehat{\omega_{k_i}} \wedge ... \wedge \omega_p.
\end{aligned}$$

We now need to shift ω_α to its correct place, which involves $|\beta - \alpha| - \gamma$ swaps, where γ is as above. Thus:

$$X_{\alpha,\beta} \cdot \star(\omega_{k_1} \wedge ... \wedge \omega_{k_i}) = \quad (-1)^{\varepsilon + |\beta-\alpha| - \gamma}\, \omega_1 \wedge ... \wedge \widehat{\omega_{k_1}} \wedge .. \wedge \widehat{\omega_\beta} \wedge .. \wedge \widehat{\omega_{k_i}} \wedge .. \wedge \omega_p$$

Finally, observe that $\varepsilon + |\beta - \alpha| - \gamma \equiv \gamma + \varepsilon' \mod (2)$ and so

$$\star \cdot X_{\alpha,\beta} = X_{\alpha,\beta} \cdot \star \text{ as operators.}$$

4. The proof is analogous when $\alpha \notin \{k_1, k_2, ..., k_i\}$, and $\beta \in \{k_1, k_2, ..., k_i\}$

□

We now proceed to computing the complexes and subsequently the cohomology groups for $p \geq 3$ in the case $l = 0$ and $p \geq 5$ in the case $l = 1$.

Remark 5.7. We only consider $p \geq 3$ in the $l = 0$ case as for $p = 2$ we have that the standard representation $V = \mathbb{C}^2$ of $\mathfrak{so}(2)$ is irreducible. If $p = 1$ then $\mathfrak{k} = \mathfrak{so}(1) = \{0\}$ and so every vector in $\mathcal{F} \otimes \bigwedge^i \mathfrak{p}^*$ will be in $\left[\mathcal{F} \otimes \bigwedge^i \mathfrak{p}^*\right]^K$ as every vector is trivially $\mathfrak{k}/K$ invariant. In the $l = 1$ case we will additionally exclude the cases when $p = 3$ or $p = 4$. We exclude $p = 3$ as the decomposition of $\mathrm{Sym}^a(V) \otimes \mathrm{Sym}^b(V)$ which was given in Section 4 only holds for $p \geq 4$, and we will make use of this decomposition in this section. Finally, we exclude $p = 4$ because $\bigwedge^2 \mathfrak{p}$ is reducible in this case. Of course, the results given in this section can be made to fit these excluded cases, nevertheless we exclude them as the analogous results for the excluded cases either do not hold for the reasons given in this section or the set of $\mathfrak{k}-$invariant vectors given is not an exhaustive list, e.g. in the case where $p = 1$, where we observe that every vector is $\mathfrak{k}-$invariant.

5.1 $l = 0$

Observe that in the case $l = 0$ we have

$$C^i_0 := C^i\big(\mathfrak{g}, K; \mathcal{F} \otimes \mathrm{Sym}^0(V)\big) = \Big[\mathcal{F} \otimes \bigwedge^i \mathfrak{p}^* \otimes 1\Big]^K \cong \Big[\mathcal{F} \otimes \bigwedge^i \mathfrak{p}^*\Big]^K.$$

Thus, throughout this subsection we will omit "$\otimes\, \mathrm{Sym}^0(V)$" and refer to the complex as $C^\bullet(\mathfrak{g}, K; \mathcal{F})$, or simply $C^\bullet$. Recall from Section 2 that the differential is given by

$$d = \sum_{\alpha=1}^{p} \left(-\frac{\partial^2}{\partial z_\alpha \partial z_{p+1}} + z_\alpha z_{p+1} \right) \otimes A(\omega_\alpha)$$

where $A(\omega_\alpha)$ denotes left exterior multiplication on $\bigwedge^\bullet \mathfrak{p}^*$ by $\omega_\alpha \in \mathfrak{p}^*$.

Remark 5.8. The results from this subsection are already known, and can be found in [3, Theorem 4.1.1, $k = 1$]. We will reproduce the results here using a different method.

5.1.1 Computing $C^\bullet(\mathfrak{g}, K; \mathcal{F})$

We will now compute complex $C^\bullet(\mathfrak{g}, K; \mathcal{F})$, our method will be as follows:

1. To begin with, we will only consider the spaces C^i where $i \leq \frac{p}{2}$. This is due to the fact that $\bigwedge^i \mathfrak{p}^*$ is an irreducible representation of $\mathfrak{so}(p)$ for $i < \frac{p}{2}$, and we completely understand its decomposition in the case $i = \frac{p}{2}$.

2. We will restrict ourselves to $C^i_+ := \Big[\mathcal{F}_+ \otimes \bigwedge^i \mathfrak{p}^*\Big]^{SO(p)}$, where $\mathcal{F}_+$ is the restriction of $\mathcal{F}$ to the first p variables, that is, $\mathcal{F}_+ = \mathbb{C}[z_1, ..., z_p] \subset \mathcal{F}$.

3. We observe that the action of $\mathfrak{k}$ on $\mathcal{F}$ preserves the degree of the polynomials, this allows us to consider each degree in turn.

4. The observation in step 3 leads us to define $\mathcal{F}_{n+}$, the restriction of F_+ to homogeneous polynomials of degree n only. We then observe that as $\mathfrak{k}$ representations we have $\mathcal{F}_{n+} \cong \mathrm{Sym}^n(V_+)$, where $V_+ = \langle e_1, ..., e_p \rangle$ is the standard representation of $\mathfrak{so}(p)$.

5. We then define $C^i|_{n+} := \Big[\mathcal{F}_{n+} \otimes \bigwedge^i \mathfrak{p}^*\Big]^{SO(p)} \cong \Big[\mathrm{Sym}^n(V_+) \otimes \bigwedge^i \mathfrak{p}^*\Big]^{SO(p)}$

6. We can compute the dimension of $C^i|_{n+}$, since we have

$$\begin{aligned} \dim\left(C^i|_{n+}\right) : &= \dim\left(\Big[\mathcal{F}_{n+} \otimes \bigwedge^i \mathfrak{p}^*\Big]^{SO(p)}\right) = \dim\left(\Big[\mathcal{F}_{n+} \otimes \bigwedge^i \mathfrak{p}^*\Big]^{\mathfrak{so}(p)}\right) \\ &= \dim\left(\mathrm{Hom}_{\mathfrak{so}(p)}\left(\bigwedge^i V_+,\ \mathrm{Sym}^n(V_+)\right)\right) \end{aligned}$$

   where we have used that $\mathfrak{p} \cong V_+$.[4]

[4] We may interchange $SO(p)$ and $\mathfrak{so}(p)$ here because $K = SO(p)$ is connected. [2, Chapter 1, Section 5.1].

7. After obtaining the dimension we find the $\mathfrak{k}-$invariant vectors in C^i_+, this will use results from [4], as well as some ad-hoc methods. Fortunately, as we will see, the dimensions are always small.

8. We then observe that $\mathfrak{k}$ acts trivially on $z_{p+1} \in \mathcal{F}\backslash\mathcal{F}_+$, so in particular multiplication by z_{p+1} commutes with the action of $\mathfrak{k}$ on $\mathcal{F}$. Therefore, we can easily reincorporate it to return to our original space C^i.

9. Finally, we will use the isomorphism $\star : \bigwedge^i \mathfrak{p}^* \to \bigwedge^{p-i} \mathfrak{p}^*$ to find the spaces C^i for $i > \frac{p}{2}$.

Recall from Section 4 that we have the following decomposition of $\mathrm{Sym}^n(V_+)$ as a representation of $\mathfrak{so}(p)$:

$$\mathrm{Sym}^n(V_+) = \mathbb{S}_{[n]}(V_+) \oplus \mathbb{S}_{[n-2]}(V_+) \oplus ... \oplus \mathbb{S}_{[n-2k]}(V_+)$$

where k is the largest integer such that $k \leq \frac{n}{2}$, and $\mathbb{S}_{[d]}(V_+)$ is the irreducible representation with highest weight dL_1.

Computing $\mathbf{C^0}\big(\mathfrak{g}, \mathbf{K}; \mathcal{F}\big)$

We have

$$C^0 = \Big[\mathcal{F} \otimes \bigwedge^0 \mathfrak{p}\Big]^K = \big[\mathcal{F} \otimes \mathbb{1}\big]^K$$

where $\mathbb{1}$ is the trivial one dimensional representation. We begin by restricting ourselves to

$$C^0|_{n+} = \big[\,\mathrm{Sym}^n(V_+) \otimes \mathbb{1}\big]^{SO(p)}.$$

We now find the dimension of this space for each $n \in \mathbb{Z}_{\geq 0}$.

Lemma 5.9. *We have:*

$$\begin{aligned}\dim\big(C^0|_{n+}\big) &= \dim\Big(\mathrm{Hom}_{\mathfrak{so}(p)}\big(\mathbb{1}, \mathrm{Sym}^n(V_+)\big)\Big) \\ &= \#\ \textit{copies of}\ \mathbb{1}\ \textit{in the representation}\ \mathrm{Sym}^n(V_+) \\ &= \begin{cases} 1 & \textit{if } n \textit{ is even} \\ 0 & \textit{if } n \textit{ is odd}\end{cases}\end{aligned}$$

Proof. $\mathrm{Sym}^n(V_+)$ has decomposition

$$\mathrm{Sym}^n(V_+) = \mathbb{S}_{[n]}(V_+) + ... + \mathbb{S}_{[n-2l]}(V_+)$$

where l is the largest integer such that $l \leq \frac{p}{2}$. We know $\mathbb{1}$ is the irreducible representation with highest weight 0 and so $\mathbb{1} = \mathbb{S}_{[0]}(V_+)$. Thus, $\mathrm{Sym}^n(V_+)$ contains a single copy of $\mathbb{1}$ if n is even and no copies of $\mathbb{1}$ if n is odd, as required. $\square$

We now state and prove the following result which will be used repeatedly throughout this section.

Lemma 5.10. *Let* $r^2 := \sum_{i=1}^{p} z_i^2$*, then for all* $X_{\alpha,\beta} \in \mathfrak{k}$*,* $f \in \mathcal{F} = \mathbb{C}[z_1, ..., z_p, z_{p+1}]$ *we have*

$$\omega(X_{\alpha,\beta}) \cdot r^2 f = r^2 \omega(X_{\alpha,\beta}) \cdot f.$$

That is, (polynomial) multiplication by r^2 *commutes with the action of* $\mathfrak{k}$ *on* $\mathcal{F}$*.*

Proof. Recall that

$$\omega(X_{\alpha,\beta}) = -z_\alpha \frac{\partial}{\partial z_\beta} + z_\beta \frac{\partial}{\partial z_\alpha}.$$

Thus,

$$\begin{aligned}
\omega(X_{\alpha,\beta}) \cdot r^2 f &= \left(-z_\alpha \frac{\partial}{\partial z_\beta} + z_\beta \frac{\partial}{\partial z_\alpha} \right) \cdot (r^2 f) = -z_\alpha \frac{\partial}{\partial z_\beta} \cdot (r^2 f) + z_\beta \frac{\partial}{\partial z_\alpha} \cdot (r^2 f) \\
&= -2 z_\alpha z_\beta f - r^2 z_\alpha \frac{\partial}{\partial z_\beta} f + 2 z_\alpha z_\beta f + r^2 z_\beta \frac{\partial}{\partial z_\alpha} f \\
&= r^2 \left(-z_\alpha \frac{\partial}{\partial z_\beta} f + z_\beta \frac{\partial}{\partial z_\alpha} f \right) = r^2 \omega(X_{\alpha,\beta}) \cdot f.
\end{aligned}$$

□

Remark 5.11. Observe that by the above result we have that

$$p \otimes 1 \in \left[\mathcal{F}_+ \otimes \mathbb{1} \right]^{SO(p)} \iff r^2 p \otimes 1 \in \left[\mathcal{F}_+ \otimes \mathbb{1} \right]^{SO(p)}.$$

Furthermore, if $\deg(p) = n$ then $\deg(r^2 p) = n + 2$. Thus, it is no surprise that the dimension of $C^0|_{n+}$ depends on the parity of n. In fact, this will be the case for all $C^i|_{n+}$.

Lemma 5.12.

$$C^0_+ = \bigoplus_{k=0}^{\infty} \langle (r^2)^k \cdot 1 \otimes 1 \rangle$$

where $r^2 := \sum_{i=1}^{p} z_i^2$*, as above, and* $\langle \cdot \rangle$ *denotes the* $\mathbb{C}-$*linear span.*

Proof. We being by checking that $1 \otimes 1$ is $\mathfrak{k}$-invariant. Indeed for $\alpha, \beta \in \{1, 2, ..., p\}$ we have

$$\begin{aligned}
X_{\alpha\beta} \cdot (1 \otimes 1) &= \omega(X_{\alpha\beta}) \cdot 1 \otimes 1 \;+\; 1 \otimes ad^*_{X_{\alpha\beta}} \cdot 1 \\
&= \left(-z_\alpha \frac{\partial}{\partial z_\beta} + z_\beta \frac{\partial}{\partial z_\alpha} \right) \cdot 1 \otimes 1 \;+\; 1 \otimes ad^*_{X_{\alpha\beta}} \cdot 1 \\
&= 0 \otimes 1 + 1 \otimes 0 = 0.
\end{aligned}$$

Now, as $\dim \left(C^0|_{0+} \right) = 1$, we must have $C^0|_{0+} = \langle 1 \otimes 1 \rangle$. Now using the previous remark we have $C^0|_{2k+} = \langle (r^2)^k \cdot 1 \otimes 1 \rangle$ and so we conclude that $C^0_+ = \bigoplus_{k=0}^{\infty} \langle (r^2)^k \cdot 1 \otimes 1 \rangle$. □

Now, to reintroduce $z_{p+1} \in \mathcal{F} \backslash \mathcal{F}_+$, we observe that the action of $\mathfrak{k}$ on $\mathcal{F}$ commutes with z_{p+1}, due to the fact that

$$\frac{\partial}{\partial z_i} z_{p+1} = z_{p+1} \frac{\partial}{\partial z_i}$$

for $i \in \{1, 2, ..., p\}$, thus we may simply add it back in. We immediately obtain the final result.

Lemma 5.13. *We have*

$$C^0\big(\mathfrak{g}, K; \mathcal{F}\big) = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot 1 \otimes 1 \rangle.$$

Computing $\mathbf{C^1}\big(\mathfrak{g}, \mathbf{K}; \mathcal{F}\big)$

We compute C^1 in exactly the same way as we computed C^0. We have

$$C^1 = \Big[\mathcal{F} \otimes \bigwedge^1 \mathfrak{p}^*\Big]^K = \big[\mathcal{F} \otimes \mathfrak{p}^*\big]^K.$$

Once again, we restrict ourselves to

$$C^1|_{n+} = \big[\operatorname{Sym}^n(V_+) \otimes \mathfrak{p}^*\big]^{SO(p)}$$

and find the dimension of this space for each $n \in \mathbb{Z}_{\geq 0}$.

Lemma 5.14. *We have*

$$\begin{aligned} \dim\big(C^1|_{n+}\big) &= \dim\Big(\operatorname{Hom}_{\mathfrak{so}(p)}\big(\mathfrak{p}, \operatorname{Sym}^n(V_+)\big)\Big) \\ &= \begin{cases} 1 & \textit{if } n \textit{ is odd} \\ 0 & \textit{if } n \textit{ is even} \end{cases} \end{aligned}$$

Proof. We begin by recalling that as $\mathfrak{k}$ representations we have $\mathfrak{p} \cong V_+$. We also have

$$\operatorname{Sym}^n(V_+) = \mathbb{S}_{[n]}(V_+) + ... + \mathbb{S}_{[n-2l]}(V_+).$$

Now, for $p \geq 3$, V_+ is an irreducible representation with highest weight L_1 and so $V_+ = \mathbb{S}_{[1]}(V_+)$. Thus, $\operatorname{Sym}^n(V_+)$ contains one copy of $\mathfrak{p} \cong V_+$ if n is odd and no copies of $\mathfrak{p}$ if n is even, as required. $\square$

Lemma 5.15.

$$C_+^1 = \bigoplus_{k=0}^{\infty} \langle (r^2)^k \cdot \varphi \rangle$$

where $\varphi = \sum_{\alpha=1}^{p} z_\alpha \otimes \omega_\alpha$.

Remark 5.16. $[\varphi] = \left[\sum_{\alpha=1}^{p} z_\alpha \otimes \omega_\alpha\right] \in H^1(\mathfrak{g}, K; \mathcal{F})$ is a "Kudla-Millson class." See [4, Lemma 7.6, q=1]. As the name suggests, this will turn out to be a representative for a non-zero cohomology class.

Proof. We begin by checking that φ is $\mathfrak{k}$-invariant. Indeed

$$\begin{aligned}
X_{\alpha,\beta} \cdot \varphi &= \sum_\gamma \omega(X_{\alpha,\beta}) \cdot z_\gamma \otimes \omega_\gamma + \sum_\gamma z_\gamma \otimes ad^*_{X_{\alpha,\beta}} \cdot \omega_\gamma \\
&= \sum_\gamma \left(-z_\alpha \frac{\partial}{\partial z_\beta} + z_\beta \frac{\partial}{\partial z_\alpha}\right) \cdot z_\gamma \otimes \omega_\gamma + \sum_\gamma z_\gamma \otimes ad^*_{X_{\alpha,\beta}} \cdot \omega_\gamma \\
&= -z_\alpha \otimes \omega_\beta + z_\beta \otimes \omega_\alpha + z_\alpha \otimes \omega_\beta - z_\beta \otimes \omega_\alpha \\
&= 0
\end{aligned}$$

Thus, $C^1|_{1+} = \langle \varphi \rangle$. Once again, as r^2 commutes with the action of $\mathfrak{k}$ we conclude that

$$C^1_+ = \bigoplus_{k=0}^{\infty} \langle (r^2)^k \cdot \varphi \rangle.$$

□

We may now reintroduce z_{p+1}, exactly as we did when computing C^0, to obtain the final result:

Lemma 5.17.

$$C^1\big(\mathfrak{g}, K; \mathcal{F}\big) = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot \varphi \rangle.$$

Computing $\mathbf{C^i\big(\mathfrak{g}, K; \mathcal{F}\big)}$ for $\mathbf{2 \leq i \leq \frac{p}{2}}$

We have

$$C^i = \left[\mathcal{F} \otimes \bigwedge^i \mathfrak{p}\right]^K.$$

We will show that, in fact, $C^i = \{0\}$ for $2 \leq i \leq \frac{p}{2}$. We begin with the following lemma.

Lemma 5.18. *Let $i \in \mathbb{Z}$ such that $2 \leq i \leq \frac{p}{2}$, then:*

$$\dim\left(C^i|_{n+}\right) = \dim\left(\operatorname{Hom}_{\mathfrak{so}(p)}\left(\bigwedge^i \mathfrak{p}, \operatorname{Sym}^n(V_+)\right)\right) = 0$$

Proof. We know that $\operatorname{Sym}^n(V_+)$ has decomposition

$$\operatorname{Sym}^n(V_+) = \mathbb{S}_{[n]}(V_+) + \ldots + \mathbb{S}_{[n-2l]}(V_+).$$

Furthermore, $\mathfrak{p} \cong V_+$ and so $\bigwedge^i \mathfrak{p} \cong \bigwedge^i V_+$. Now, from Section 4 we know that for $i < \frac{p}{2}$ (note the strict inequality), $\bigwedge^i V_+$ is an irreducible representation with highest weight $L_1 + L_2 + ... + L_i$ and so $\bigwedge^i V_+ = S_{(1,1,...,1)}(V_+)$. Thus, for $2 \leq i < \frac{p}{2}$ we have that

$$\mathrm{Sym}^n(V_+) = S_{[n]}(V_+) + ... + S_{[n-2l]}(V_+) = S_{(n,0,...,0)}(V_+) + ... + S_{(n-2l,0,...,0)}(V_+)$$

does not contain a copy of $\bigwedge^i V_+$. If p is even then we also must consider the case where $i = \frac{p}{2}$. In this case $\bigwedge^i V_+$ decomposes into the direct sum of two irreducible representations, one with highest weight $(1, 1, ..., 1)$, and the other with highest weight $(1, ..., 1, -1)$. As the decomposition of $\mathrm{Sym}^n(V_+)$ does not contain either of the irreducible representations with these highest weights, there are no $\mathfrak{so}(p)$-homomorphisms from $\bigwedge^i \mathfrak{p}^*$ to $\mathrm{Sym}^n(V_+)$, and so $\dim\left(C^i|_{n+}\right) = 0$. □

Therefore, we have $C^i_+ = \{0\}$. Reintroducing the z_{p+1} term gives us the result.

Lemma 5.19.

$$C^i(\mathfrak{g}, K; \mathcal{F}) = \{0\}.$$

Computing $\mathbf{C^{p-i}}(\mathfrak{g}, \mathbf{K}; \mathcal{F})$ for $\mathbf{2 \leq i \leq \frac{p}{2}}$

We now compute the spaces

$$C^{p-i} = \left[\mathcal{F} \otimes \bigwedge^{p-i} \mathfrak{p}^*\right]^K$$

for $2 \leq i < \frac{p}{2}$. To do this we use the $\mathfrak{so}(p)$ isomorphism

$$\star : \bigwedge^i \mathfrak{p}^* \to \bigwedge^{p-i} \mathfrak{p}^*.$$

We begin by restricting to the space

$$C^{p-i}|_{n+} = \left[\mathrm{Sym}^n(V_+) \otimes \bigwedge^{p-i} \mathfrak{p}^*\right]^{SO(p)}$$

and finding its dimension. We have:

Lemma 5.20. *Let $i \in \mathbb{Z}_{\geq 0}$ such that $2 \leq i \leq \frac{p}{2}$, then*

$$\begin{aligned}\dim\left(C^{p-i}|_{n+}\right) &= \dim\left(\mathrm{Hom}_{\mathfrak{so}(p)}\left(\bigwedge^{p-i} \mathfrak{p},\, \mathrm{Sym}^n(V_+)\right)\right) \\ &= \dim\left(\mathrm{Hom}_{\mathfrak{so}(p)}\left(\bigwedge^{i} \mathfrak{p}, \mathrm{Sym}^n(V_+)\right)\right) \\ &= 0\end{aligned}$$

Proof. This follows from the fact that $\star$ is a $\mathfrak{so}(p)$-isomorphism and $\dim\left(C^i|_{n+}\right) = 0$. □

Thus we have $C^{p-i}_+ = \{0\}$. We reintroduce the z_{p+1} term to get the following result:

Lemma 5.21.

$$C^{p-i}(\mathfrak{g}, K; \mathcal{F}) = \{0\}$$

Computing $\mathbf{C^{p-1}}(\mathfrak{g}, \mathbf{K}; \mathcal{F})$

We will once again make use of the $\mathfrak{so}(p)$-isomorphism, $\star$. Firstly, we have:

$$C^{p-1} = \left[\mathcal{F} \otimes \bigwedge^{p-1} \mathfrak{p}^*\right]^K.$$

We restrict to

$$C^{p-1}|_{n+} = \left[\,\mathrm{Sym}^n(V_+) \otimes \bigwedge^{p-1} \mathfrak{p}^*\right]^{SO(p)}$$

and compute its dimension:

Lemma 5.22.

$$\begin{aligned}
\dim\left(C^{p-1}|_{n+}\right) &= \dim\left(\,\mathrm{Hom}_{\mathfrak{so}(p)}\left(\bigwedge^{p-1} \mathfrak{p},\, \mathrm{Sym}^n(V_+)\right)\right) \\
&= \dim\left(\,\mathrm{Hom}_{\mathfrak{so}(p)}\left(\mathfrak{p}, \mathrm{Sym}^n(V_+)\right)\right) \\
&= \begin{cases} 1 & \textit{if } n \textit{ is odd} \\ 0 & \textit{if } n \textit{ is even} \end{cases}
\end{aligned}$$

Proof. This follows immediately using $\star$ and our result for $\dim\left(C^1|_{n+}\right)$. □

In fact, $\star$ not only gives us the dimension of the spaces $C^{p-1}|_{n+}$, but it also gives us the $\mathfrak{k}$-invariant vectors. We have:

Lemma 5.23.

$$C_+^{p-1} = \bigoplus_{k=0}^{\infty} \langle (r^2)^k \cdot \star(\varphi)\rangle$$

where $\star(\varphi) := \sum_{\alpha=1}^{p} z_\alpha \otimes \star(\omega_\alpha) = \sum_{\alpha=1}^{p} (-1)^{\alpha-1}\, z_\alpha \otimes \omega_1 \wedge ... \wedge \widehat{\omega_\alpha} \wedge ... \wedge \omega_p.$

Proof. Firstly, $\star(\varphi)$ is $\mathfrak{k}$-invariant since $\star$ is a $\mathfrak{so}(p)-$isomorphism. Indeed:

$$X_{\alpha,\beta} \cdot \star(\varphi) = \star(X_{\alpha,\beta} \cdot \varphi) = 0.$$

Since $\dim\left(C^{p-1}|_{1+}\right) = 1$ we have

$$C^{p-1}|_{1+} = \langle\, \star(\varphi)\,\rangle.$$

Now, r^2 commutes with the action of $\mathfrak{k}$ on $\mathcal{F}$ so $r^2 \cdot \star(\varphi)$ is also $\mathfrak{k}$-invariant. We conclude that

$$C_+^{p-1} = \bigoplus_{k=0}^{\infty} \langle (r^2)^k \cdot \star(\varphi)\rangle.$$

□

Finally, we reintroduce z_{p+1} to obtain:

Lemma 5.24.

$$C^{p-1}(\mathfrak{g}, K; \mathcal{F}) = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot \star(\varphi)\rangle.$$

Computing $\mathbf{C^p}\big(\mathfrak{g}, \mathbf{K}; \mathcal{F}\big)$

Finally, we compute the space C^p. We let $\Omega := \omega_1 \wedge \omega_2 \wedge ... \wedge \omega_p$. We have

$$C^p = \Big[\mathcal{F} \otimes \bigwedge\nolimits^p \mathfrak{p}^*\Big]^K = \big[\mathcal{F} \otimes \langle\Omega\rangle\big]^K.$$

Once again, we restrict to

$$C^p|_{n+} = \big[\,\mathrm{Sym}^n(V_+) \otimes \langle\Omega\rangle\big]^{SO(p)}$$

and find its dimension. We have:

Lemma 5.25.

$$\dim\big(C^p|_{n+}\big) = \begin{cases} 1 & \textit{if } n \textit{ is even} \\ 0 & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. The result follows from the fact that $\bigwedge^p \mathfrak{p}^* = \langle\Omega\rangle \cong \mathbb{1}$ and the result for $\dim\big(C^0|_{n+}\big)$. $\square$

Lemma 5.26.

$$C^p_+ = \bigoplus_{k=0}^{\infty} \langle (r^2)^k \cdot 1 \otimes \Omega\rangle.$$

Proof. Firstly, $1 \otimes \Omega$ is $\mathfrak{k}$-invariant. Indeed, this follows from the fact that $\star$ is a $\mathfrak{so}(p)-$isomorphism. Thus we have

$$C^p|_{0+} = \langle 1 \otimes \Omega\rangle.$$

Then, as r^2 commutes with the action of $\mathfrak{k}$ on $\mathcal{F}$, we conclude that

$$C^p_+ = \bigoplus_{k=0}^{\infty} \langle (r^2)^k \cdot 1 \otimes \Omega\rangle.$$

$\square$

Finally, we reintroduce the z_{p+1} term to obtain:

Lemma 5.27.

$$C^p\big(\mathfrak{g}, K; \mathcal{F}\big) = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot 1 \otimes \Omega\rangle.$$

5.1.2 Computing the cohomology groups $H^\bullet(\mathfrak{g}, K; \mathcal{F})$

We now compute the cohomology groups associated to the complex $C^\bullet(\mathfrak{g}, K; \mathcal{F})$. Recall that:

$$H^i := H^i(\mathfrak{g}, K; \mathcal{F}) = {\ker\big(d : C^i \to C^{i+1}\big)}\Big/{\mathrm{im}\big(d : C^{i-1} \to C^i\big)}$$

where d denotes the differential which is given by

$$d = \sum_{\alpha=1}^{p} \omega(X_{\alpha,p+1}) \otimes A(\omega_\alpha) = \sum_{\alpha=1}^{p} \left(-\frac{\partial^2}{\partial z_\alpha \partial z_{p+1}} + z_\alpha z_{p+1} \right) \otimes A(\omega_\alpha)$$

where A denotes exterior multiplication by $\omega_\alpha \in \mathfrak{p}^*$ on $\bigwedge^i \mathfrak{p}^*$.

As mentioned in Section 2, the cohomology groups are in fact $\mathfrak{sl}(2)-$modules, where the action on the cohomology classes is induced by the action of $\mathfrak{sl}(2)$ on $\mathcal{F}$, that is, if $\Psi = \sum_i p_i \otimes \Omega_i \in C^\bullet(\mathfrak{g}, K; \mathcal{F})$, and $X \in \mathfrak{sl}(2)$, then X acts on $[\Psi]$ as follows:

$$X \cdot [\Psi] = [X \cdot \Psi] = \left[\sum_i \omega(X) \cdot p_i \otimes \Omega_i \right].$$

As the action of $\mathfrak{sl}(2)$ and $\mathfrak{o}(p,1)$ on $\mathcal{F}$ commute, one sees that the action of $\mathfrak{sl}(2)$ commutes with both the differential and the action of $\mathfrak{k} \subset \mathfrak{o}(p,1)$, thus the action of $\mathfrak{sl}(2)$ on the cohomology groups makes sense and is well defined. Using the structure theorem (Theorem 3.10) from Section 3, we will describe the $\mathfrak{sl}(2)-$module structure of each of the cohomology groups $H^i(\mathfrak{g}, K; \mathcal{F})$.

Remark 5.28. As $C^i(\mathfrak{g}, K; \mathcal{F}) = \{0\}$ for $i \neq 0, 1, p-1, p$, we have

$$H^i(\mathfrak{g}, K; \mathcal{F}) = \{0\}$$

for $i \neq 0, 1, p-1, p$. Thus, we need only consider the cases when $i \in \{0, 1, p-1, p\}$.

Computing $\mathbf{H^0}(\mathfrak{g}, \mathbf{K}; \mathcal{F})$

Recall that

$$C^0(\mathfrak{g}, K; \mathcal{F}) = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot 1 \otimes 1 \rangle.$$

We now compute differential of each of these vectors.

$$\begin{aligned} & d\big(z_{p+1}^l \cdot (r^2)^k \cdot 1 \otimes 1\big) \\ &= \sum_{\alpha=1}^{p} \omega(X_{\alpha,p+1}) \cdot z_{p+1}^l \cdot (r^2)^k \cdot 1 \otimes \omega_\alpha \\ &= \sum_{\alpha=1}^{p} \Big(-\frac{\partial^2}{\partial z_\alpha z_{p+1}} + z_\alpha z_{p+1} \Big) \cdot z_{p+1}^l \cdot (r^2)^k \cdot 1 \otimes \omega_\alpha \end{aligned}$$

$$= \sum_{\alpha=1}^{p} -\frac{\partial}{\partial z_\alpha}\Big(l z_{p+1}^{l-1} \cdot (r^2)^k \cdot 1\Big) \otimes \omega_\alpha + \sum_{\alpha=1}^{p} z_\alpha z_{p+1}^{l+1} \cdot (r^2)^k \cdot 1 \otimes \omega_\alpha$$
$$= z_{p+1}^{l+1} \cdot (r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \omega_\alpha \;-\; 2lk \cdot z_{p+1}^{l-1} \cdot (r^2)^{k-1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \omega_\alpha.$$

From this computation we can see that no combination

$$\sum_i z_{p+1}^{l_i} \cdot (r^2)^{k_i} \cdot 1 \otimes 1$$

where $l_i, k_i \in \mathbb{Z}_{\geq 0}$ will be in the kernel of d, i.e. $\ker\big(d : C^0 \to C^1\big) = \{0\}$. Therefore, we obtain:

Theorem 5.29.
$$H^0\big(\mathfrak{g}, K; \mathcal{F}\big) = \{0\}.$$

Computing $\mathbf{H^1}\big(\mathfrak{g}, \mathbf{K}; \mathcal{F}\big)$

Recall that

$$C^1(\mathfrak{g}, K; \mathcal{F}) = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot \varphi \rangle$$

where $\varphi = \sum_{\alpha=1}^{p} z_\alpha \otimes \omega_\alpha$. We compute the differentials of these vectors:

$$\begin{aligned}
&d\big(z_{p+1}^l \cdot (r^2)^k \cdot \varphi\big) \\
&= \sum_{\alpha=1}^{p} \sum_{\beta=1}^{p} \omega(X_{\alpha,p+1}) \cdot z_{p+1}^l \cdot (r^2)^k \cdot z_\beta \otimes \omega_\alpha \wedge \omega_\beta \\
&= \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{p} \Big(-\frac{\partial^2}{\partial z_\alpha z_{p+1}} + z_\alpha z_{p+1}\Big) \cdot z_{p+1}^l \cdot (r^2)^k \cdot z_\beta \otimes \omega_\alpha \wedge \omega_\beta \\
&= \sum_{\substack{\alpha,i=\beta \\ \alpha\neq\beta}}^{p} -\frac{\partial}{\partial z_\alpha} \cdot l z_{p+1}^{l-1} \cdot (r^2)^k \cdot z_\beta \otimes \omega_\alpha \wedge \omega_\beta + \sum_{\substack{\alpha,i=\beta \\ \alpha\neq\beta}}^{p} z_\alpha \cdot z_{p+1}^{l+1} \cdot (r^2)^k \cdot z_\beta \otimes \omega_\alpha \wedge \omega_\beta \\
&= \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{p} -l z_{p+1}^{l-1} \cdot k(r^2)^{k-1} \cdot 2z_\alpha \cdot z_\beta \otimes \omega_\alpha \wedge \omega_\beta + \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{p} z_\alpha \cdot z_{p+1}^{l+1} \cdot (r^2)^k \cdot z_\beta \otimes \omega_\alpha \wedge \omega_\beta \\
&= -2lk \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{p} z_{p+1}^{l-1} \cdot (r^2)^{k-1} \cdot z_\alpha z_\beta \otimes \omega_\alpha \wedge \omega_\beta + \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{p} z_{p+1}^{l+1} \cdot (r^2)^k \cdot z_\alpha z_\beta \otimes \omega_\alpha \wedge \omega_\beta \\
&= 0 + 0 = 0, \text{ since } \omega_\alpha \wedge \omega_\beta = -\omega_\beta \wedge \omega_\alpha,\ z_\alpha z_\beta = z_\beta z_\alpha, \text{ and we sum over all } \alpha, \beta.
\end{aligned}$$

Thus, we have

$$\ker\left(d : C^1 \to C^2\right) = C^1(\mathfrak{g}, K; \mathcal{F}) = \bigoplus_{k=0}^{\infty} \bigoplus_{l=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot \varphi \rangle.$$

Moreover, from our calculations for H^0 we have for $l, k \geq 0$

$$d\big(z_{p+1}^l \cdot (r^2)^k \cdot 1 \otimes 1\big) = z_{p+1}^{l+1} \cdot (r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \omega_\alpha - 2lk \cdot z_{p+1}^{l-1} \cdot (r^2)^{k-1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \omega_\alpha$$

which implies that in $H^1(\mathfrak{g}, K; \mathcal{F})$ we have the following cohomology relation:

$$\left[z_{p+1}^{l+1} \cdot (r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \omega_\alpha\right] = 2lk\left[z_{p+1}^{l-1} \cdot (r^2)^{k-1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \omega_\alpha\right].$$

This gives the following following characterisation of $H^1(\mathfrak{g}, K; \mathcal{F})$:

Theorem 5.30.

$$H^1(\mathfrak{g}, K; \mathcal{F}) = \bigoplus_{k=0}^{\infty} \left\langle \left[(r^2)^k \cdot \varphi\right] \right\rangle, \quad \textit{where} \;\; \varphi = \sum_{\alpha=1}^{p} z_\alpha \otimes \omega_\alpha.$$

Proof. Given $\left[z_{p+1}^l \cdot (r^2)^k \cdot \varphi\right] \in H^1$, use the cohomology relation

$$\left[z_{p+1}^{l+1} \cdot (r^2)^k \cdot \varphi\right] = 2lk\left[z_{p+1}^{l-1} \cdot (r^2)^{k-1} \cdot \varphi\right]$$

repeatedly until at least one of z_{p+1}^l or $(r^2)^k$ have vanished, or we are left with a class which is a multiple of $\left[z_{p+1} \cdot (r^2)^k \cdot \varphi\right]$. If we obtain $\left[z_{p+1}^a \cdot \varphi\right]$, $a > 0$, as a result of applying the cohomology relation, then observe that $d\big(z_{p+1}^{a-1} \cdot 1 \otimes 1\big) = z_{p+1}^a \cdot \varphi$ and so $\left[z_{p+1}^a \cdot \varphi\right] = [0]$. If we are left with the class of the form $\left[z_{p+1} \cdot (r^2)^k \cdot \varphi\right]$, then $\mathrm{d}\left((r^2)^k \otimes 1\right) = z_{p+1} \cdot (r^2)^k \cdot \varphi$, and so $\left[z_{p+1} \cdot (r^2)^k \cdot \varphi\right] = [0]$. $\square$

We now discuss the $\mathfrak{sl}(2)-$module structure of $H^1(\mathfrak{g}, K; \mathcal{F})$ and its submodules.

Theorem 5.31. $\bigoplus_{k=0}^{\infty} \left\langle \left[(r^2)^k \cdot \varphi\right] \right\rangle$ *has* $\mathfrak{sl}(2)-$*module structure* $\big([\circ)$, *that is,* $\bigoplus_{k=0}^{\infty} \left\langle \left[(r^2)^k \cdot \varphi\right] \right\rangle$ *is an infinite dimensional lowest-weight* $\mathfrak{sl}(2, \mathbb{C})$ *module. Moreover, it has lowest weight* $\dfrac{(p+1)i}{2}$ *and lowest weight vector* $[\varphi]$.

Remark 5.32. One may include the information about the lowest weight and lowest weight vector on a " dot diagram" as follows, where the values above the dots represents the weights, and the cohomology classes below the dots represents an element from the corresponding weight space.

$$\bigoplus_{k=0}^{\infty}\left\langle\left[(r^2)^k\cdot\varphi\right]\right\rangle \;\simeq\; \left[\quad \begin{array}{ccccc} \frac{(p+1)i}{2} & & & \frac{(p+4k+1)i}{2} & \\ \circ & \cdots & \circ & \circ & \cdots \\ [\varphi] & & & \left[(r^2)^k\cdot\varphi\right] & \end{array}\right.$$

Proof. Using the actions of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathcal{F}$ given in Section 2 we obtain that:

$$\begin{aligned} L\cdot[\varphi]=[L\cdot\varphi] &= \left[\Big(-\frac{1}{2}\sum_{\alpha=1}^{p}\frac{\partial^2}{\partial z_\alpha^2}+\frac{1}{2}z_{p+1}^2\Big)\cdot\sum_{\beta=1}^{p}z_\beta\otimes\omega_\beta\right] \\ &= \left[\frac{1}{2}z_{p+1}^2\sum_{\beta=1}^{p}z_\beta\otimes\omega_\beta\right]=\left[d\Big(\frac{1}{2}z_{p+1}\otimes 1\Big)\right]=[0] \end{aligned}$$

$$\begin{aligned} H\cdot[\varphi]=[H\cdot\varphi] &= \left[i\Big(\sum_{\alpha=1}^{p}z_\alpha\frac{\partial}{\partial z_\alpha}-z_{p+1}\frac{\partial}{\partial z_{p+1}}+\frac{(p-1)}{2}\Big)\cdot\sum_{i=1}^{p}z_i\otimes\omega_i\right] \\ &= i\left[\sum_{i=1}^{p}z_i\otimes\omega_i+\frac{(p-1)}{2}\sum_{i=1}^{p}z_i\otimes\omega_i\right]=\frac{(p+1)i}{2}\cdot[\varphi] \end{aligned}$$

Thus, we have shown that $[\varphi]$ vanishes under the action of L and therefore is a lowest-weight vector. Moreover $[\varphi]$ has weight $\dfrac{(p+1)i}{2}$. Next, one easily sees that

$$\left[R\cdot(r^2)^{k-1}\cdot\varphi\right]=\frac{1}{2}\left[(r^2)^k\cdot\varphi\right].$$

All that remains is to check what happens when we apply L to $\left[(r^2)^k\cdot\varphi\right]$. We have

$$\begin{aligned} \left[L\cdot(r^2)^k\cdot\varphi\right] &= \left[\Big(-\frac{1}{2}\sum_{\alpha=1}^{p}\frac{\partial^2}{\partial z_\alpha^2}+\frac{1}{2}z_{p+1}^2\Big)\cdot(r^2)^k\cdot\sum_{\beta=1}^{p}z_\beta\otimes\omega_\beta\right] \\ &= \left[-\frac{1}{2}\sum_{\alpha,\beta=1}^{p}\frac{\partial^2}{\partial z_\alpha^2}\Big((r^2)^k\cdot z_\beta\Big)\otimes\omega_\beta\right]+\underbrace{\left[\frac{1}{2}z_{p+1}^2\cdot(r^2)^k\cdot\sum_{\alpha=1}^{p}z_\alpha\otimes\omega_\alpha\right]}_{\text{Use cohomology relation.}} \\ &= -\frac{1}{2}\Big(4k(k-1)+2kp+5k\Big)\left[(r^2)^{k-1}\cdot\varphi\right] \\ &= -\frac{k}{2}\Big(4k+2p+1\Big)\left[(r^2)^{k-1}\cdot\varphi\right] \end{aligned}$$

This is non-zero for all $k>0$ since $4k+2p+1\neq 0$ for all $k>0$. $\square$

Remark 5.33. If $[\Psi]$ is a cohomology class such that $L \cdot [\Psi] = [0]$ then we say that $[\Psi]$ is a *holomorphic* class. This is due to a correspondence between the cohomology of the Weil representation and holomorphic automorphic forms on suitable quotients of ${SL(2,\mathbb{R})}/{U(n)}$. In particular, the Kudla-Millson class, $[\varphi]$, is a holomorphic class. No further details of this correspondence will be discussed in this report. For more information, see [3, Chapter 1], [6], [4].

Computing $\mathbf{H^{p-1}}(\mathfrak{g}, \mathbf{K}; \mathcal{F})$

Recall that we have

$$C^{p-1}(\mathfrak{g}, K; \mathcal{F}) = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot \star(\varphi) \rangle$$

where

$$\star(\varphi) := \sum_{\alpha=1}^{p} z_\alpha \otimes \star(\omega_\alpha) = \sum_{\alpha=1}^{p} (-1)^{\alpha-1}\ z_\alpha \otimes \omega_1 \wedge ... \wedge \widehat{\omega_\alpha} \wedge ... \wedge \omega_p.$$

Let $\Omega := \omega_1 \wedge ... \wedge \omega_p$, then the differential of these vectors is:

$$d\Big(z_{p+1}^l \cdot (r^2)^k \cdot \star(\varphi)\Big) = -(2lk + lp) \cdot z_{p+1}^{l-1} \cdot (r^2)^k \cdot 1 \otimes \Omega + z_{p+1}^{l+1} \cdot (r^2)^{k+1} \cdot 1 \otimes \Omega.$$

We see from this calculation that $\ker \big(d : C^{p-1} \to C^p\big) = \{0\}$. Thus we obtain:

Theorem 5.34.

$$H^{p-1}(\mathfrak{g}, K; \mathcal{F}) = \{0\}.$$

Computing $\mathbf{H^{p}}(\mathfrak{g}, \mathbf{K}; \mathcal{F})$

Recall that

$$C^{p}(\mathfrak{g}, K; \mathcal{F}) = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot 1 \otimes \Omega \rangle.$$

Now, as $p = \dim \mathfrak{p}$ we have:

Lemma 5.35.

$$d\Big(z_{p+1}^l \cdot (r^2)^k \cdot 1 \otimes \Omega\Big) = 0$$

for all $l,k \in \mathbb{Z}_{\geq 0}$.

Proof.

$$\begin{aligned} d\Big(z_{p+1}^l \cdot (r^2)^k \cdot 1 \otimes \omega_1 \wedge ... \wedge \omega_p\Big) &= \sum_{\alpha=1}^{p} \omega(X_{\alpha,p+1}) \cdot z_{p+1}^l \cdot (r^2)^k \cdot 1 \otimes \omega_\alpha \wedge \omega_1 \wedge ... \wedge \omega_p \\ &= \sum_{\alpha=1}^{p} \omega(X_{\alpha,p+1}) \cdot z_{p+1}^l \cdot (r^2)^k \cdot 1 \otimes 0 \ = \ 0. \end{aligned}$$

$\square$

From our calculations for H^{p-1} we have for $l, k \geq 0$:

$$d\Big(z_{p+1}^{l} \cdot (r^2)^k \cdot \star(\varphi)\Big) = -(2lk + lp) \cdot z_{p+1}^{l-1} \cdot (r^2)^k \cdot 1 \otimes \Omega + z_{p+1}^{l+1} \cdot (r^2)^{k+1} \cdot 1 \otimes \Omega.$$

Thus, in $H^p(\mathfrak{g}, K; \mathcal{F})$ we have the following cohomology relation:

$$\Big[z_{p+1}^{l+1} \cdot (r^2)^{k+1} \cdot 1 \otimes \Omega\Big] = (2lk + lp) \cdot \Big[z_{p+1}^{l-1} \cdot (r^2)^k \cdot 1 \otimes \Omega\Big].$$

This gives us the following characterisation of $H^p(\mathfrak{g}, K; \mathcal{F})$:

Theorem 5.36.

$$H^p(\mathfrak{g}, K; \mathcal{F}) = \bigoplus_{l=0}^{\infty} \langle \Big[z_{p+1}^{l} \otimes \Omega\Big] \rangle \bigoplus \bigoplus_{k=1}^{\infty} \langle \Big[(r^2)^k \otimes \Omega\Big] \rangle.$$

Proof. Given a class $\Big[z_{p+1}^{l} \cdot (r^2)^k \cdot 1 \otimes \Omega\Big]$, use the cohomology relation repeatedly until either z_{p+1} or r^2 vanishes or we are left with $\Big[z_{p+1} \cdot (r^2)^k \cdot 1 \otimes \Omega\Big], k \geq 1$. Observe that $d\Big(\big(r^2\big)^{k-1} \cdot \star(\phi)\Big) = z_{p+1} \cdot (r^2)^k \cdot 1 \otimes \Omega$ and so $\Big[z_{p+1}(r^2)^k \cdot 1 \otimes \Omega\Big] = \big[0\big]$. □

We now describe the $\mathfrak{sl}(2)-$module structure of $H^p(\mathfrak{g}, K; \mathcal{F})$ and its submodules.

Theorem 5.37. $\bigoplus_{l=0}^{\infty} \langle \Big[z_{p+1}^{2l+1} \otimes \Omega\Big] \rangle$ *has* $\mathfrak{sl}(2, \mathbb{C})-$*module structure:*

1. $\big(\circ\,]\,\circ\,]\big)$ *if p is odd.*
2. $\big(\circ\,]\big)$ *if p is even.*

Moreover, if p is even then $\big[z_{p+1} \otimes \Omega\big]$ is the highest weight vector with weight $\dfrac{(p-3)i}{2}$. If p is odd then $\big[z_{p+1} \otimes \Omega\big]$ and $\big[z_{p+1}^{p} \otimes \Omega\big]$ are the highest weight vectors.

Proof. We have:

$$R \cdot \Big[z_{p+1} \otimes \Omega\Big] = \Big[\frac{1}{2} r^2 \cdot z_{p+1} \otimes \Omega\Big] = \Big[d\Big(\frac{1}{2} \cdot \star(\varphi)\Big)\Big] = \big[0\big].$$

$$H \cdot \Big[z_{p+1} \otimes \Omega\Big] = \Big[-i \cdot z_{p+1} \otimes \Omega + \frac{(p-1)i}{2} \cdot z_{p+1} \otimes \Omega\Big] = \frac{(p-3)i}{2}\Big[z_{p+1} \otimes \Omega\Big].$$

Furthermore, one easily sees that

$$\Big[L \cdot z_{p+1}^{2l+1} \otimes \Omega\Big] = \frac{1}{2}\Big[z_{p+1}^{2(l+1)+1} \otimes \Omega\Big].$$

Finally, we must check what happens when R is applied to $\left[z_{p+1}^{2l+1}\otimes\Omega\right]$ for $l\geq 1$.

$$\left[R\cdot z_{p+1}^{2l+1}\otimes\Omega\right]=\Bigg[\underbrace{\frac{1}{2}\cdot r^2\cdot z_{p+1}^{2l+1}\otimes\Omega}_{\text{Use cohomology relation}}-\frac{1}{2}\frac{\partial^2}{\partial z_{p+1}^2}\left(z_{p+1}^{2l+1}\right)\otimes\Omega\Bigg]$$
$$=l\Big(p-(2l+1)\Big)\cdot\left[z_{p+1}^{2l-1}\otimes\Omega\right]$$

Thus, when $2l+1=p$, i.e. when p is odd, we have that $R\cdot[z_{p+1}^p\otimes\Omega]=[0]$. $\square$

Theorem 5.38. $\bigoplus_{l=0}^{\infty}\left\langle\left[z_{p+1}^{2l}\otimes\Omega\right]\right\rangle\bigoplus\bigoplus_{k=1}^{\infty}\left\langle\left[(r^2)^k\otimes\Omega\right]\right\rangle$ *has* $\mathfrak{sl}(2)-$*module structure:*

1. $(\circ)$ *if p is odd.*
2. $(\circ]\circ)$ *if p is even.*

Moreover, if p is even then $\left[z_{p+1}^p\otimes\Omega\right]$ *is the highest weight vector, that is,*

$$R\cdot\left[z_{p+1}^p\otimes\Omega\right]=[0].$$

Proof. The proof of this result is essentially the same as the previous one. To begin, we easily observe that

$$\left[R\cdot(r^2)^{k-1}\otimes\Omega\right]=\frac{1}{2}\left[(r^2)^k\otimes\Omega\right]\text{ and }\left[L\cdot z_{p+1}^{2(l-1)}\otimes\Omega\right]=\frac{1}{2}\left[z_{p+1}^{2l}\otimes\Omega\right].$$

Thus, we only have to check what happens when L is applied to $\left[(r^2)^k\otimes\Omega\right]$ and when R is applied to $\left[z_{p+1}^{2l}\otimes\Omega\right]$. We have:

$$Ł\cdot\left[(r^2)^k\otimes\Omega\right]=\Bigg[-\frac{1}{2}\sum_{\alpha=1}^{p}\frac{\partial^2}{\partial z_\alpha^2}\Big((r^2)^k\Big)\otimes\Omega+\underbrace{\frac{1}{2}z_{p+1}^2\cdot(r^2)^k\otimes\Omega}_{\text{Use cohomology relation.}}\Bigg]$$
$$=-\frac{1}{2}\Big(2k-1\Big)\Big(2k-(2-p)\Big)\cdot\left[(r^2)^{k-1}\otimes\Omega\right]$$

We observe that $\Big(2k-1\Big)\Big(2k-(2-p)\Big)$ has no positive integer solutions k, thus we have that $L\cdot\left[(r^2)^k\otimes\Omega\right]\neq\left[0\right]$ for all $k\geq 1$. Finally we have:

$$\left[R\cdot z_{p+1}^{2l}\otimes\Omega\right]=\Bigg[\underbrace{\frac{1}{2}\cdot r^2\cdot z_{p+1}^{2l}\otimes\Omega}_{\text{Use cohomology relation}}-\frac{1}{2}\cdot 2l(2l-1)\cdot z_{p+1}^{2l-2}\otimes\Omega\Bigg]$$
$$=-\frac{1}{2}\big(2l-p\big)\big(2l-1\big)\cdot\left[z_{p+1}^{2l-2}\otimes\Omega\right]$$

Thus, when $2l=p$, i.e. when p is even, we have that $\left[R\cdot z_{p+1}^p\otimes\Omega\right]=\left[0\right]$. $\square$

5.2 $l = 1$

We will now consider the complex

$$C_1^\bullet := C^\bullet(\mathfrak{g}, K; \mathcal{F} \otimes V) = \Big[\mathcal{F} \otimes \bigwedge\nolimits^\bullet \mathfrak{p}^* \otimes V\Big]^K.$$

Recall from Section 2 that the differential is given by $d = d_s + d_v$ with

$$d_s = \sum_{\alpha=1}^{p} \omega(X_{\alpha,p+1}) \otimes A(\omega_\alpha) \otimes 1, \quad \text{and} \quad d_v = \sum_{\alpha=1}^{p} 1 \otimes A(\omega_\alpha) \otimes \rho(X_{\alpha,p+1})$$

where

$$\omega(X_{\alpha,p+1}) = -\frac{\partial^2}{\partial z_\alpha \partial z_{p+1}} + z_\alpha z_{p+1}$$

and where ρ is the derived action of $\mathfrak{p} \subset \mathfrak{g} = \mathrm{Lie}(G)$ on V. Explicitly, one finds that for all $X_{\alpha,\beta} \in \mathfrak{k}$ and $X_{\delta,p+1} \in \mathfrak{p}$ we have

$$\rho(X_{\alpha,\beta}) \cdot e_\gamma = \begin{cases} e_\beta & \text{if } \gamma = \alpha \\ -e_\alpha & \text{if } \gamma = \beta \\ 0 & \text{otherwise} \end{cases}, \qquad \rho(X_{\delta,p+1}) \cdot e_\gamma = \begin{cases} e_\delta & \text{if } \gamma = p+1 \\ e_{p+1} & \text{if } \gamma = \delta \\ 0 & \text{otherwise.} \end{cases}$$

We will compute $C_1^\bullet = C^\bullet\big(\mathfrak{o}(p,1), K; \mathcal{F} \otimes V\big)$ in the case $p \geq 5$. Recall from Section 4 that we have the following decomposition of $\mathrm{Sym}^a(V_+) \otimes \mathrm{Sym}^b(V_+)$ as $\mathfrak{so}(p)$ representations, where V_+ is the standard representation of $\mathfrak{k} \cong \mathfrak{so}(p)$:

$$\mathrm{Sym}^a(V_+) \otimes \mathrm{Sym}^b(V_+) = \bigoplus_{l=0}^{\lfloor \frac{a}{2} \rfloor} \Bigg(\bigoplus_{k \geq \frac{b-(a-2l)}{2}}^{k \leq \frac{b}{2}} \ \bigoplus_{0 \leq \zeta \leq b-2k} \ \bigoplus_{\alpha \in Y(a-2l, b-2k, \zeta)} \Gamma_\alpha \\ \oplus \bigoplus_{k=0}^{k < \frac{b-(a-2l)}{2}} \ \bigoplus_{0 \leq \zeta \leq a-2l} \ \bigoplus_{\alpha \in Y(b-2k, a-2l, \zeta)} \Gamma_\alpha \Bigg).$$

In the case $b = 1$ the formula takes the much simpler form

$$\mathrm{Sym}^a(V_+) \otimes V_+ = \bigoplus_{l=0}^{l < \frac{a}{2}} \Bigg(\bigoplus_{0 \leq \zeta \leq 1} \ \bigoplus_{\alpha \in Y(a-2l, 1, \zeta)} \Gamma_\alpha \Bigg) \oplus \bigoplus_{l = \frac{a}{2}} \Gamma_{(1)}.$$

5.2.1 Computing $C^\bullet\big(\mathfrak{g}, K; \mathcal{F} \otimes V\big)$

Our method for computing this complex will very similar to our method for computing the complex in the case $l = 0$, it will be as follows:

1. To begin with, we will only consider the spaces C_1^i where $i \leq \frac{p}{2}$.

2. We will restrict to $C_{1+}^i := \Big[\mathcal{F}_+ \otimes \bigwedge^i \mathfrak{p}^* \otimes V_+\Big]^{SO(p)}$ where $\mathcal{F}_+ = \mathbb{C}[z_1, ..., z_p]$ is the restriction of $\mathcal{F}$ to the first p variables and $V_+ = \langle e_1, ..., e_p\rangle$.

3. We observe that the action of $\mathfrak{k}$ on $\mathcal{F}$ preserves the degree of the polynomials, therefore allowing us to consider each degree in turn.

4. Step 3 leads us to define $\mathcal{F}_{n+}$, the restriction of F_+ to homogeneous polynomials of degree n. We then observe as $\mathfrak{k}$ representations we have $\mathcal{F}_{n+} \cong \mathrm{Sym}^n(V_+)$.

5. We define $C_1^i|_{n+} := \Big[\mathcal{F}_{n+} \otimes \bigwedge^i \mathfrak{p}^* \otimes V_+\Big]^{SO(p)} = \Big[\mathrm{Sym}^n(V_+) \otimes \bigwedge^i \mathfrak{p}^* \otimes V_+\Big]^{SO(p)}$.

6. We can compute the dimension of $C_1^i|_{n+}$ since

$$\dim\big(C_1^i|_{n+}\big) = \dim\Big(\mathrm{Hom}_{\mathfrak{so}(p)}\Big(\bigwedge^i V_+,\ \mathrm{Sym}^n(V_+) \otimes V_+\Big)\Big).$$

7. After obtaining the dimension (which will turn out to always be small) we find the $\mathfrak{k}-$invariant vectors in $C_1^i|_+$, this step will use results from [6] as well as some ad-hoc methods.

8. We re-incorporate $z_{p+1} \in \mathcal{F}\backslash\mathcal{F}_+$ exactly as we did in the $l = 0$ case. We will also devise a method to re-introduce $e_{p+1} \in V\backslash V_+$.

9. For $i > \frac{p}{2}$, we use the isomorphism $\star : \bigwedge^i \mathfrak{p}^* \to \bigwedge^{p-i} \mathfrak{p}^*$ to compute C_1^i for $i > \frac{p}{2}$.

Computing $\mathbf{C^0}\big(\mathfrak{g}, \mathbf{K}; \mathcal{F} \otimes \mathbf{V}\big)$

We have

$$C_1^0 := C^0\big(\mathfrak{g}, K; \mathcal{F} \otimes V\big) = \big[\mathcal{F} \otimes \mathbb{1} \otimes V\big]^K.$$

We begin by restricting ourselves to

$$C_1^0|_{n+} = \big[\mathrm{Sym}^n(V_+) \otimes \mathbb{1} \otimes V_+\big]^{SO(p)}$$

and computing the dimension of $C_1^0|_{n+}$ for each $n \in \mathbb{Z}_{\geq 0}$.

Lemma 5.39.

$$\begin{aligned}
\dim\big(C_1^0|_{n+}\big) &= \dim\Big(\mathrm{Hom}_{\mathfrak{so}(p)}\Big(\mathbb{1}, \mathrm{Sym}^n(V_+) \otimes V_+\Big)\Big) \\
&= \#\ \textit{copies of } \mathbb{1} \textit{ in } \mathrm{Sym}^n(V_+) \otimes V_+ \\
&= \begin{cases} 1 & \textit{if } n \textit{ odd} \\ 0 & \textit{if } n \textit{ even} \end{cases}
\end{aligned}$$

Proof. Recall that the decomposition of $\mathrm{Sym}^n(V_+)\otimes V+$ is:

$$\mathrm{Sym}^n(V_+)\otimes V_+ = \bigoplus_{l=0}^{l<\frac{n}{2}}\left(\bigoplus_{0\leq\zeta\leq 1}\ \bigoplus_{\alpha\in Y(n-2l,1,\zeta)}\Gamma_\alpha\right)\bigoplus\bigoplus_{l=\frac{n}{2}}\Gamma_{(1)}.$$

Now, $\mathbb{1}$ is the irreducible representation of $\mathfrak{so}(m)$ with highest weight 0, so we have $\mathbb{1}=\Gamma_{(0)}$. If the decomposition contains any of copies of $\Gamma_{(0)}$ then we certainly require $n-2l+1-2\zeta=0$. This implies that n must be odd.

Now, suppose n is odd, we see that $l=\frac{n-1}{2}$ and $\zeta=1$ give $\alpha\in Y(1,1,1)$. Moreover, we see that this gives the only copy of $\Gamma_{(0)}$ present in the decomposition. $\square$

Lemma 5.40. *We have*

$$C^0_{1+}=\bigoplus_{k=0}^{\infty}\big\langle (r^2)^k\cdot\varphi_{0,1}\big\rangle.$$

where $\varphi_{0,1}=\displaystyle\sum_{\alpha=1}^{p} z_\alpha\otimes 1\otimes e_\alpha$.

Remark 5.41. $[\varphi_{0,1}]=\left[\displaystyle\sum_{\alpha=1}^{p} z_\alpha\otimes 1\otimes e_\alpha\right]\in H^1_1$ is a so-called *Funke-Millson* class. See [6, Chapter 6.2]. We will see later that this gives a non-zero cohomology class in H^1_1.

Proof. We begin by showing that $\varphi_{0,1}$ is $\mathfrak{k}-$invariant. Indeed:

$$\begin{aligned}X_{\alpha,\beta}\cdot\varphi_{0,1}&=\sum_{\gamma=1}^{p}\omega(X_{\alpha,\beta})\cdot z_\gamma\otimes 1\otimes e_\gamma+\sum_{\gamma=1}^{p} z_\gamma\otimes 1\otimes\rho(X_{\alpha,\beta})\cdot e_\gamma\\&=-z_\alpha\otimes 1\otimes e_\beta+z_\beta\otimes 1\otimes e_\alpha+z_\alpha\otimes 1\otimes e_\beta-z_\beta\otimes 1\otimes e_\alpha=0\end{aligned}$$

Next, since r^2 commutes with the action of $\mathfrak{k}$ all vectors of the form $(r^2)^k\cdot\varphi_{0,1}$ are $\mathfrak{k}-$invariant. Finally, since $\dim\left(C^0_1|_{n+}\right)=1$ when n is odd we know that we have found all $\mathfrak{k}-$invariant vectors. $\square$

Now, exactly as in the case $l=0$, we have that $\mathfrak{k}$ acts trivially on z_{p+1} so we can simply reintroduce it to obtain:

$$\big[\mathcal{F}\otimes 1\otimes V_+\big]^K=\bigoplus_{l,k=0}^{\infty}\big\langle z^l_{p+1}\cdot(r^2)^k\cdot\varphi_{0,1}\big\rangle.$$

We now wish to reintroduce the e_{p+1} term, this is not as straightforward as reintroducing z_{p+1}, but is nevertheless very doable after the following observation. This observation will apply for all C^i_1 so we discuss it now in the general case. We observe that for any

$$\sum_j p_j\otimes\Omega_j\otimes v_j\in\mathcal{F}\otimes\bigwedge^i\mathfrak{p}^*\otimes V,$$

one may write it as

$$\underbrace{\sum_j p_j \otimes \Omega_j \otimes v_{j+}}_{\in \mathcal{F}\otimes\bigwedge^i \mathfrak{p}^*\otimes V_+} + \underbrace{\sum_j p_j \otimes \Omega_j \otimes \lambda_j e_{p+1}}_{\in \mathcal{F}\otimes\bigwedge^i \mathfrak{p}^*\otimes\langle e_{p+1}\rangle}$$

by linearity, where $\lambda_j \in \mathbb{C}$ is a scalar. Now, as $\mathfrak{k}$ acts trivially on e_{p+1} we have:

$$X_{\alpha,\beta} \cdot \sum_j p_j \otimes \Omega_j \otimes \lambda_j e_{p+1} = \sum_j \omega(X_{\alpha,\beta}) \cdot p_j \otimes \Omega_j \otimes \lambda_j e_{p+1} + \sum_j p_j \otimes X_{\alpha,\beta} \cdot \Omega_j \otimes \lambda_j e_{p+1}.$$

This observation gives us the following general result which we will use repeatedly throughout this subsection:

Lemma 5.42. *Let J be an indexing set, then*

$$\sum_{j\in J} p_j \otimes \Omega_j \otimes \lambda_j e_{p+1} \in \left[\mathcal{F} \otimes \bigwedge^i \mathfrak{p}^* \otimes V\right]^K$$
$$\iff \sum_{j\in J} \lambda_j p_j \otimes \Omega_j \in \left[\mathcal{F} \otimes \bigwedge^i \mathfrak{p}^*\right]^K.$$

Furthermore, for each $j \in J$, let $v_j = v_{j+} + \lambda_j e_{p+1}$, where $v_{j_+} \in V_+$, then

$$\sum_j p_j \otimes \Omega_j \otimes v_j \in \left[\mathcal{F} \otimes \bigwedge^i \mathfrak{p}^* \otimes V\right]^K$$
$$\iff \sum_j p_j \otimes \Omega_j \otimes v_{j+} \in \left[\mathcal{F} \otimes \bigwedge^i \mathfrak{p}^* \otimes V_+\right]^K$$
$$\text{and } \sum_j \lambda_j p_j \otimes \Omega_j \in \left[\mathcal{F} \otimes \bigwedge^i \mathfrak{p}^*\right]^K.$$

Proof. This result follows from the fact that both V_+ and $\langle e_{p+1}\rangle$ are closed under the $\mathfrak{k}$-action, so both terms in the decomposition must be independently $\mathfrak{k}$-invariant. $\square$

We know from the $l = 0$ case that $[\mathcal{F} \otimes 1]^K = C^0(\mathfrak{g}, K; \mathcal{F}) = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot 1 \otimes 1\rangle$ and so $\left[\mathcal{F} \otimes 1 \otimes \langle e_{p+1}\rangle\right]^K = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot 1 \otimes 1 \otimes e_{p+1}\rangle$. Now, using Lemma 5.42 with $i = 0$ gives us the result for the space $C_1^0 = C^1(\mathfrak{g}, K; \mathcal{F} \otimes V)$:

Lemma 5.43.

$$C_1^0 = \left[\mathcal{F} \otimes 1 \otimes V\right]^K = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot \varphi_{0,1}\rangle \bigoplus \bigoplus_{l,k=0}^{\infty} \left\langle z_{p+1}^l \cdot (r^2)^k \cdot 1 \otimes 1 \otimes e_{p+1}\right\rangle$$

where $\varphi_{0,1} = \sum_{\alpha=1}^{p} z_\alpha \otimes 1 \otimes e_\alpha$.

Computing $\mathbf{C^1}\big(\mathfrak{g}, \mathbf{K}; \mathcal{F} \otimes \mathbf{V}\big)$

We have

$$C_1^1 := C^1(\mathfrak{g}, K; \mathcal{F} \otimes V) = \big[\mathcal{F} \otimes \mathfrak{p}^* \otimes V\big]^K$$

We proceed in exactly the same way as in the C_0^1 case. We begin by restricting ourselves to

$$C_1^1|_{n+} := \big[\,\mathrm{Sym}^n(V_+) \otimes \mathfrak{p}^* \otimes V_+\big]^{SO(p)}$$

and computing its dimension. We have:

Lemma 5.44.

$$\dim\big(C_1^1|_{n+}\big) = \dim\Big(\,\mathrm{Hom}_{\mathfrak{so}(p)}\big(V_+, \mathrm{Sym}^n(V_+) \otimes V_+\big)\Big)$$
$$= \begin{cases} 0 \ \textit{if } n \ \textit{odd} \\ 1 \ \textit{if } n = 0 \\ 2 \ \textit{if } n > 0 \ \textit{and even} \end{cases}$$

Proof. The decomposition of $\mathrm{Sym}^n(V_+) \otimes V_+$ is:

$$\mathrm{Sym}^n(V_+) \otimes V_+ = \bigoplus_{l=0}^{l<\frac{n}{2}} \left(\bigoplus_{0 \le \zeta \le 1} \bigoplus_{\alpha \in Y(n-2l,1,\zeta)} \Gamma_\alpha \right) \ \bigoplus \ \bigoplus_{l=\frac{n}{2}} \Gamma_{(1)}$$

We have $V_+ \cong \Gamma_{(1)}$. If $\Gamma_{(1)}$ is present in the decomposition then we certainly require $n - 2l + 1 - 2\zeta = 1$. This implies that n must be even. Now suppose n is even, then we immediately obtain a copy of $\Gamma_{(1)}$ from the $l = \frac{n}{2}$ part of the direct sum. Now if $n > 0$ we obtain another copy of $\Gamma_{(1)}$ through $l = \frac{n}{2} - 1$ and $\zeta = 1$, of course, this copy is only obtainable for $n \ge 2$. This completes the proof. $\square$

Lemma 5.45.

$$C_{1+}^1 = \bigoplus_{k=0}^{\infty} \langle (r^2)^k \cdot f \rangle \ \bigoplus \ \bigoplus_{k=0}^{\infty} \langle (r^2)^k \cdot \varphi_{1,1} \rangle$$

where $f = \displaystyle\sum_{\alpha=1}^{p} 1 \otimes \omega_\alpha \otimes e_\alpha$, $\ \varphi_{1,1} = \displaystyle\sum_{\alpha,\beta=1}^{p} z_\alpha z_\beta \otimes \omega_\alpha \otimes e_\beta$.

Remark 5.46. $\big[\varphi_{1,1}\big]$ is a Funke-Millson class, see [6, Chapter 6.2].

We now reintroduce z_{p+1} and obtain:

$$\big[\mathcal{F} \otimes \mathfrak{p}^* \otimes V_+\big]^K = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot f \rangle \ \bigoplus \ \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot \varphi_{1,1} \rangle.$$

Finally, we re-introduce e_{p+1}. We know from the $l = 0$ case that

$$\big[\mathcal{F} \otimes \mathfrak{p}^*\big]^K = \bigoplus_{k=0}^{\infty} \bigoplus_{l=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot \varphi \rangle.$$

where $\varphi = \sum_{\alpha=1}^{p} z_\alpha \otimes e_\alpha$. Thus, by Lemma 5.42, we obtain the following result:

Lemma 5.47.

$$C_1^1 = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot f\rangle \bigoplus \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot \varphi_{1,1}\rangle \bigoplus \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot \varphi_{1,0}\rangle$$

where $f = \sum_{\alpha=1}^{p} 1 \otimes \omega_\alpha \otimes e_\alpha, \quad \varphi_{1,1} = \sum_{\alpha,\beta=1}^{p} z_\alpha z_\beta \otimes \omega_\alpha \otimes e_\beta, \quad \varphi_{1,0} = \sum_{\alpha=1}^{p} z_\alpha \otimes \omega_\alpha \otimes e_{p+1}.$

Computing $\mathbf{C^2}(\mathfrak{g}, \mathbf{K}; \mathcal{F} \otimes \mathbf{V})$

We have

$$C_1^2 := C^2(\mathfrak{g}, K; \mathcal{F} \otimes V) = \Big[\mathcal{F} \otimes \bigwedge^2 \mathfrak{p}^* \otimes V\Big]^K .$$

Once again we begin by restricting ourselves to

$$C_1^2|_{n+} = \Big[\mathrm{Sym}^n(V_+) \otimes \bigwedge^2 \mathfrak{p}^* \otimes V_+\Big]^{SO(p)}$$

and computing the dimension.

Lemma 5.48.

$$\begin{aligned} \dim\left(C_1^2|_{n+}\right) &= \dim\left(\mathrm{Hom}_{\mathfrak{so}(p)}\left(\bigwedge^2 V_+,\, \mathrm{Sym}^n(V_+) \otimes V_+\right)\right) \\ &= \begin{cases} 1 & \textit{if } n \textit{ odd} \\ 0 & \textit{if } n \textit{ even} \end{cases} \end{aligned}$$

Proof. We have $\bigwedge^2 V_+ \cong \Gamma_{(1,1)}$, so we are looking for copies of the irreducible representation $\Gamma_{(1,1)}$ in the decomposition of $\mathrm{Sym}^n(V_+) \otimes V_+$. Recall that we have:

$$\mathrm{Sym}^n(V_+) \otimes V_+ = \bigoplus_{l=0}^{l<\frac{n}{2}} \left(\bigoplus_{0 \le \zeta \le 1} \bigoplus_{\alpha \in Y(n-2l,1,\zeta)} \Gamma_\alpha \right) \bigoplus \bigoplus_{l=\frac{n}{2}} \Gamma_{(1)}.$$

We require $n - 2l + 1 - 2\zeta = 2 \iff n - 2l - 2\zeta = 1 \implies n$ odd. We consider the two cases in which $n - 2l + 1 - 2\zeta = 2$ can be achieved:

- Case 1: $n - 2l = 1$, $\zeta = 0$, this gives us $\alpha \in Y(1,1,0)$ with $Y(1,1,0) = \{(2),(1,1)\}$. This case gives us a copy of $\Gamma_{(1,1)}$.
- Case 2: $n - 2l = 3$, $\zeta = 1$, this gives us $\alpha \in Y(3,1,1)$ where $Y(3,1,1) = \{(2)\}$. This case does not give us a copy of $\Gamma_{(1,1)}$.

□

Lemma 5.49.

$$C^2_{1+} = \bigoplus_{k=0}^{\infty} \Big\langle (r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha \otimes \omega_\alpha \wedge \omega_\beta \otimes e_\beta \Big\rangle.$$

Proof. We will show that

$$\sum_{\alpha,\beta=1}^{p} z_\alpha \otimes \omega_\alpha \wedge \omega_\beta \otimes e_\beta$$

is $\mathfrak{k}$-invariant. Once this is shown, the well established fact that r^2 commutes with the action of $\mathfrak{k}$ will give the result. We have:

$$\begin{aligned}
&X_{\gamma,\delta} \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha \otimes w_\alpha \wedge w_\beta \otimes e_\beta \\
=&\sum_{\alpha,\beta=1}^{p} \omega(X_{\gamma,\delta}) \cdot z_\alpha \otimes \omega_\alpha \wedge \omega_\beta \otimes e_\beta + \sum_{\alpha,\beta=1}^{p} z_\alpha \otimes ad^*_{X_{\gamma,\delta}}(\omega_\alpha \wedge \omega_\beta) \otimes e_\beta \\
+&\sum_{\alpha,\beta=1}^{p} z_\alpha \otimes \omega_\alpha \wedge \omega_\beta \otimes \rho(X_{\gamma,\delta}) e_\beta \\
=&\sum_{\beta=1}^{p} \big(z_\delta \otimes \omega_\gamma \wedge \omega_\beta \otimes e_\beta - z_\gamma \otimes \omega_\delta \wedge \omega_\beta \otimes e_\beta\big) + \sum_{\beta=1}^{p} \Big(z_\gamma \otimes \omega_\delta \wedge \omega_\beta \otimes e_\beta - z_\delta \otimes \omega_\gamma \wedge \omega_\beta \otimes e_\beta\Big) \\
+&\sum_{\alpha=1}^{p} \Big(z_\alpha \otimes \omega_\alpha \wedge \omega_\delta \otimes e_\gamma - z_\alpha \otimes \omega_\alpha \wedge \omega_\gamma \otimes e_\delta\Big) + \sum_{\alpha=1}^{p} \big(z_\alpha \otimes \omega_\alpha \wedge \omega_\gamma \otimes e_\delta - z_\alpha \otimes \omega_\alpha \wedge \omega_\delta \otimes e_\gamma\big) \\
=&0
\end{aligned}$$

To obtain the final equality, observe that the summations in the first line cancel each other, as do the summations in the second line. □

We re-introduce z_{p+1} to obtain

$$\Big[\mathcal{F} \otimes \bigwedge^2 \mathfrak{p}^* \otimes V_+\Big]^K = \bigoplus_{l,k=0}^{\infty} \big\langle z^l_{p+1} \cdot (r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha \otimes \omega_\alpha \wedge \omega_\beta \otimes e_\beta \big\rangle.$$

We now reintroduce e_{p+1}. We know from the $l = 0$ case that $\Big[\mathcal{F} \otimes \bigwedge^2 \mathfrak{p}^*\Big]^K = \{0\}$, and so by Lemma 5.42 we obtain the following result:

Lemma 5.50.

$$C^2_1 = \Big[\mathcal{F} \otimes \bigwedge^2 \mathfrak{p}^* \otimes V\Big]^K = \bigoplus_{l,k=0}^{\infty} \big\langle z^l_{p+1} \cdot (r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha \otimes \omega_\alpha \wedge \omega_\beta \otimes e_\beta \big\rangle.$$

Computing $\mathbf{C^i}\big(\mathfrak{g},K;\mathcal{F}\otimes V\big)$ for $\mathbf{3\leq i\leq \frac{p}{2}}$

We have

$$C_1^i := C^i(\mathfrak{g},K;\mathcal{F}\otimes V) = \Big[\mathcal{F}\otimes \bigwedge^i \mathfrak{p}^*\otimes V\Big]^K.$$

Now, we restrict to:

$$C_{1+}^i = \Big[\mathcal{F}_+\otimes \bigwedge^i \mathfrak{p}^*\otimes V_+\Big]^{SO(p)}.$$

We will show that $C_{1+}^i = \{0\}$.

Lemma 5.51. *Let $3\leq i\leq \frac{p}{2}$, then:*

$$C_{1+}^i = \{0\}.$$

Proof. We show that for all $n\in\mathbb{Z}_{\geq 0}$ we have

$$\dim\left(C_1^i|_{n+}\right) = \dim\left(\mathrm{Hom}_{\mathfrak{so}(p)}\left(\bigwedge^i V_+\,,\,\mathrm{Sym}^n(V_+)\otimes V_+\right)\right) = 0.$$

If $3\leq i<\frac{p}{2}$ (note the strict inequality) we have $\bigwedge^i V_+ = \Gamma_{\underbrace{(1,1,..,1)}_{\text{i entries}}}$. If $i=\frac{p}{2}$, then $\bigwedge^i V_+$ decomposes into two irreducible representations, one of which is $\Gamma_{\underbrace{(1,1,..,1)}_{\text{i entries}}}$ and the other is $\Gamma_{\underbrace{(1,..,1,-1)}_{\text{i entries}}}$. Thus, we need to find copies of $\Gamma_{\underbrace{(1,1,..,\pm 1)}_{\text{i entries}}}$ in the decomposition, however we know that partitions with more than 2 non-zero terms do not appear in the decomposition of $\mathrm{Sym}^n(V_+)\otimes V+$. Thus, there are no $\mathfrak{k}-$ homomorphisms from $\bigwedge^i V_+$ to $\mathrm{Sym}^n(V_+)\otimes V_+$. This completes the proof. $\square$

We now re-introduce z_{p+1} which gives:

$$\Big[\mathcal{F}\otimes \bigwedge^i \mathfrak{p}^*\otimes V_+\Big]^K = \bigoplus_{l=0}^{\infty}\left\{z_{p+1}^l\cdot 0\right\} = \{0\}.$$

Finally, we reintroduce e_{p+1} using Lemma 5.42. We know from the $l=0$ case that $\Big[\mathcal{F}\otimes \bigwedge^i \mathfrak{p}^*\Big]^K = \{0\}$, so we conclude that:

Theorem 5.52.

$$C_1^i = \{0\},\ \ \textit{for}\ \ 3\leq i\leq \frac{p}{2}.$$

Computing $\mathbf{C^{p-i}}(\mathfrak{g}, \mathbf{K}; \mathcal{F} \otimes \mathbf{V})$ for $\mathbf{3 \leq i \leq \frac{p}{2}}$

We have

$$C_1^{p-i} := C^{p-i}(\mathfrak{g}, K; \mathcal{F} \otimes V) = \left[\mathcal{F} \otimes \bigwedge^{p-i} \mathfrak{p}^* \otimes V\right]^K .$$

We now make use of the $\mathfrak{so}(p)-$isomorphism, Hodge star, which we introduced at the start of the section. Recall that the isomorphism is:

$$\begin{aligned} \star : \bigwedge^{i} \mathfrak{p}^* &\to \bigwedge^{p-i} \mathfrak{p}^* \\ \omega &\mapsto \star(\omega) \end{aligned}$$

such that $\omega \wedge \star(\omega) = \omega_1 \wedge \omega_2 \wedge ... \wedge \omega_p \in \bigwedge^p \mathfrak{p}^* \cong \mathbb{C}$. This isomorphism immediately gives us the following result:

Lemma 5.53.

$$C_1^{p-i} = \{0\}, \quad \textit{for} \;\; 3 \leq i < \frac{p}{2}.$$

Proof. Firstly, we obtain using Hodge star and our result for $\dim\left(C_1^i|_{n+}\right)$ that

$$\begin{aligned} \dim\left(C_1^{p-i}|_{n+}\right) &= \dim\left(\left[\mathcal{F} \otimes \bigwedge^{p-i} \mathfrak{p}^* \otimes V\right]^K\right) \\ &= \dim\left(\left[\mathcal{F} \otimes \bigwedge^{i} \mathfrak{p}^* \otimes V\right]^K\right) = \dim\left(C_1^i|_{n+}\right) = 0. \end{aligned}$$

for all $n \in \mathbb{Z}_{\geq 0}$ and so $C_1^{p-1}{}_+ = \{0\}$. We now re-introduce z_{p+1} and obtain $\left[\mathcal{F} \otimes \bigwedge^{p-i} \mathfrak{p}^* \otimes V_+\right]^K = \{0\}$. Moreover, from work in the $l = 0$ case we know that $\left[\mathcal{F} \otimes \bigwedge^{p-i} \mathfrak{p}^*\right]^K = \{0\}$. Thus, re-introducing the e_{p+1} term using Lemma 5.42 gives $C_1^{p-i} = \{0\}$. $\square$

Computing $\mathbf{C^{p-2}}(\mathfrak{g}, \mathbf{K}; \mathcal{F} \otimes \mathbf{V})$

Remark 5.54. We will simply state the results for $C_1^{p-2} := \left[\mathcal{F} \otimes \bigwedge^{p-2} \mathfrak{p}^* \otimes V\right]^K$, as well as for C_1^{p-1}, and C_1^p. The results stated hold since $\dim\left(C_1^{p-i}|_{n+}\right) = \dim\left(C_1^i|_{n+}\right)$ for all $n \in \mathbb{Z}_{\geq 0}$ by the isomorphism Hodge star and so $\left[\mathcal{F} \otimes \bigwedge^{p-i} \mathfrak{p}^* \otimes V_+\right]^K$ is completely determined by C_{1+}^i and the isomorphism, $\star$. Moreover $\left[\mathcal{F} \otimes \bigwedge^{p-i} \mathfrak{p}^*\right]^K$ is completely determined by $C_0^i = \left[\mathcal{F} \otimes \bigwedge^i \mathfrak{p}^*\right]^K$ and the isomorphism $\star$, as we have seen in the $l = 0$ case. Thus C_1^{p-i} is determined completely by C_1^i and $\star$ by Lemma 5.42.

Recall that:

$$C_1^2 = \left[\mathcal{F} \otimes \bigwedge^2 \mathfrak{p}^* \otimes V\right]^K = \bigoplus_{l,k=0}^{\infty} \left\langle z_{p+1}^l \cdot (r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha \otimes \omega_\alpha \wedge \omega_\beta \otimes e_\beta \right\rangle.$$

Thus, we have:

Lemma 5.55.

$$C_1^{p-2} = \left[\mathcal{F} \otimes \bigwedge^{p-2} \mathfrak{p}^* \otimes V\right]^K = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha \otimes \star(\omega_\alpha \wedge \omega_\beta) \otimes e_\beta \rangle.$$

Computing $\mathbf{C^{p-1}}(\mathfrak{g}, \mathbf{K}; \mathcal{F} \otimes \mathbf{V})$

Recall that:

$$C_1^1 = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot f \rangle \bigoplus \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot \varphi_{1,1} \rangle \bigoplus \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot \varphi_{1,0} \rangle$$

where $f = \sum_{\alpha=1}^{p} 1 \otimes \omega_\alpha \otimes e_\alpha$, $\quad \varphi_{1,1} = \sum_{\alpha,\beta=1}^{p} z_\alpha z_\beta \otimes \omega_\alpha \otimes e_\beta$, $\quad \varphi_{1,0} = \sum_{\alpha=1}^{p} z_\alpha \otimes \omega_\alpha \otimes e_{p+1}$.
Thus, we have:

Lemma 5.56.

$$C_1^{p-1} = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot \star(f) \rangle \bigoplus \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot \star(\varphi_{1,1}) \rangle \bigoplus \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot \star(\varphi_{1,0}) \rangle$$

where

$$\star(f) = \sum_{\alpha=1}^{p} 1 \otimes \star(\omega_\alpha) \otimes e_\alpha, \;\; \star(\varphi_{1,1}) = \sum_{\alpha,\beta=1}^{p} z_\alpha z_\beta \otimes \star(\omega_\alpha) \otimes e_\beta, \star(\varphi_{1,0}) = \sum_{\alpha=1}^{p} z_\alpha \otimes \star(\omega_\alpha) \otimes e_{p+1}.$$

Computing $\mathbf{C_1^p}(\mathfrak{g}, \mathbf{K}; \mathcal{F} \otimes \mathbf{V})$

Recall that:

$$C_1^0 = \left[\mathcal{F} \otimes 1 \otimes V\right]^K = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot \varphi_{0,1} \rangle \bigoplus \bigoplus_{l,k=0}^{\infty} \left\langle z_{p+1}^l \cdot (r^2)^k \cdot 1 \otimes 1 \otimes e_{p+1} \right\rangle$$

where $\varphi_{0,1} = \sum_{\alpha=1}^{p} z_\alpha \otimes 1 \otimes e_\alpha$. Thus we have:

Lemma 5.57.

$$C_1^p = \left[\mathcal{F} \otimes 1 \otimes V\right]^K = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot \star(\varphi_{0,1}) \rangle \bigoplus \bigoplus_{l,k=0}^{\infty} \left\langle z_{p+1}^l \cdot (r^2)^k \cdot 1 \otimes \Omega \otimes e_{p+1} \right\rangle$$

where $\Omega := \omega_1 \wedge \omega_2 \wedge ... \wedge \omega_p$ *and* $\star(\varphi_{0,1}) = \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha$.

5.2.2 Computing the cohomology groups $H^\bullet\big(\mathfrak{g}, K; \mathcal{F}\otimes V\big)$

We now compute the cohomology groups arising from $C^\bullet(\mathfrak{g}, K; \mathcal{F}\otimes V)$. Recall that the differential is given by

$$d = d_s + d_v$$

where

$$d_s = \sum_{\alpha=1}^{p} \omega(X_{\alpha,p+1}) \otimes A(\omega_\alpha) \otimes 1, \quad d_v = \sum_{\alpha=1}^{p} 1 \otimes A(\omega_\alpha) \otimes \rho(X_{\alpha,p+1}).$$

Exactly as in the $l = 0$ case, the cohomology groups are in fact $\mathfrak{sl}(2)-$modules, where the action is once again given by:

$$X \cdot [\Psi] = [X \cdot \Psi] = \Big[\sum_i \omega(X) \cdot p_i \otimes \Omega_i \otimes v_i\Big]$$

where $\Psi = \sum_i p_i \otimes \Omega_i \otimes v_i \in C^\bullet(\mathfrak{g}, K; \mathcal{F}\otimes V)$, and $X \in \mathfrak{sl}(2)$. Using the structure theorem (Theorem 3.10), we will describe the structure of the cohomology groups $H_1^\bullet := H^\bullet(\mathfrak{g}, K; \mathcal{F}\otimes V)$ as $\mathfrak{sl}(2)-$modules.

Remark 5.58. As $C^i\big(\mathfrak{g}, K; \mathcal{F}\otimes V\big) = \{0\}$ for $3 \le i \le p-3$, we have

$$H^i(\mathfrak{g}, K; \mathcal{F}\otimes V) = \{0\}$$

for $3 \le i \le p-3$. Thus, we need only consider the cases when $i \in \{0, 1, 2, p-2, p-1, p\}$.

Computing $\mathbf{H^0}\big(\mathfrak{g}, \mathbf{K}; \mathcal{F}\otimes \mathbf{V}\big)$

We wish to compute

$$H_1^0 := H^0\big(\mathfrak{g}, K; \mathcal{F}\otimes V\big) = {\ker\big(d : C_1^0 \to C_1^1\big)}\big/{\operatorname{im}\big(d : 0 \to C_1^0\big)} = \ker\big(d : C_1^0 \to C_1^1\big).$$

Recall that we have

$$C_1^0 = \big[\mathcal{F}\otimes 1 \otimes V\big]^K = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot \varphi_{0,1}\rangle \bigoplus \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot 1 \otimes 1 \otimes e_{p+1}\rangle$$

where $\varphi_{0,1} = \sum_{\alpha=1}^{p} z_\alpha \otimes 1 \otimes e_\alpha$. The differentials of these $\mathfrak{k}-$invariant vectors are as follows:

1. $d\Big(z_{p+1}^l \cdot (r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes 1 \otimes e_\alpha\Big)$

$$= -2lk \cdot z_{p+1}^{l-1} \cdot (r^2)^{k-1} \cdot \sum_{\alpha,\beta=1} z_\alpha z_\beta \otimes \omega_\alpha \otimes e_\beta \;-\; l \cdot z_{p+1}^{l-1} \cdot (r^2)^k \cdot \sum_{\alpha=1}^{p} 1 \otimes \omega_\alpha \otimes e_\alpha$$

$$+ z_{p+1}^{l+1} \cdot (r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha z_\beta \otimes \omega_\alpha \otimes e_\beta \;+\; z_{p+1}^{l} \cdot (r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \omega_\alpha \otimes e_{p+1}$$

2. $d\Big(z_{p+1}^{l}\cdot(r^2)^k\cdot\otimes 1\otimes e_{p+1}\Big)$.

$$= -2lk\cdot z_{p+1}^{l-1}\cdot(r^2)^{k-1}\cdot\sum_{\alpha=1}^{p} z_\alpha\otimes\omega_\alpha\otimes e_{p+1} \;+\; z_{p+1}^{l+1}\cdot(r^2)^k\cdot\sum_{\alpha=1}^{p} z_\alpha\otimes\omega_\alpha\otimes e_{p+1}$$

$$+\, z_{p+1}^{l}\cdot(r^2)^k\cdot\sum_{\alpha=1}^{p} 1\otimes\omega_\alpha\otimes e_\alpha.$$

This gives us the following result.

Theorem 5.59.

$$H^0\big(\mathfrak{g},K;\mathcal{F}\otimes V\big)=\{0\}.$$

Proof. Observe that no combination of vectors of the form

$$\sum_i\left(z_{p+1}^{l_i}\cdot(r^2)^{k_i}\cdot\sum_{\alpha=1}^{p} z_\alpha\otimes 1\otimes e_\alpha\right)\quad\text{or}\quad\sum_i z_{p+1}^{l_i}(r^2)^{k_i}\otimes 1\otimes e_{p+1}$$

will be in the kernel of d. Moreover, $d\Big(z_{p+1}^{l}\cdot(r^2)^k\cdot\sum_{\alpha=1}^{p} z_\alpha\otimes 1\otimes e_\alpha\Big)$ has a term containing a non-zero multiple of the vector $\sum_{\alpha,\beta=1}^{p} z_\alpha z_\beta\otimes\omega_\alpha\otimes e_\beta$ and $d\Big(z_{p+1}^{l}(r^2)^k\otimes 1\otimes e_{p+1}\Big)$ has no such term, therefore no combination of vectors in C_1^0 will be in the kernel of d, i.e. $\ker(d)=\{0\}$. ☐

Computing $\mathbf{H^1}\big(\mathfrak{g},\mathbf{K};\mathcal{F}\otimes\mathbf{V}\big)$

Recall that

$$C_1^1=\bigoplus_{l,k=0}^{\infty}\langle z_{p+1}^{l}\cdot(r^2)^k\cdot f\rangle\ \bigoplus\ \bigoplus_{l,k=0}^{\infty}\langle z_{p+1}^{l}\cdot(r^2)^k\cdot\varphi_{1,1}\rangle\ \bigoplus\ \bigoplus_{l,k=0}^{\infty}\langle z_{p+1}^{l}\cdot(r^2)^k\cdot\varphi_{1,0}\rangle$$

where $f=\sum_{\alpha=1}^{p}1\otimes\omega_\alpha\otimes e_\alpha$, $\varphi_{1,1}=\sum_{\alpha,\beta=1}^{p} z_\alpha z_\beta\otimes\omega_\alpha\otimes e_\beta$, $\varphi_{1,0}=\sum_{\alpha=1}^{p} z_\alpha\otimes\omega_\alpha\otimes e_{p+1}$.
The differentials of these vectors are as follows:

1. $$d\Big(z_{p+1}^{l}\cdot(r^2)^k\cdot\sum_{\alpha=1}^{p}1\otimes\omega_\alpha\otimes e_\alpha\Big)=-\,2lk\cdot z_{p+1}^{l-1}\cdot(r^2)^{k-1}\cdot\sum_{\alpha,\beta=1}^{p} z_\alpha\otimes\omega_\alpha\wedge\omega_\beta\otimes e_\beta$$
$$+\,z_{p+1}^{l+1}\cdot(r^2)^k\cdot\sum_{\alpha,\beta=1}^{p} z_\alpha\otimes\omega_\alpha\wedge\omega_\beta\otimes e_\beta$$

2. $$d\Big(z_{p+1}^{l}\cdot(r^2)^k\cdot\sum_{\alpha,\beta=1}^{p} z_\alpha z_\beta\otimes\omega_\alpha\otimes e_\beta\Big)=l\cdot z_{p+1}^{l-1}\cdot(r^2)^k\cdot\sum_{\alpha,\beta=1}^{p} z_\alpha\otimes\omega_\alpha\wedge\omega_\beta\otimes e_\beta$$

3. $d\Big(z_{p+1}^l \cdot (r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \omega_\alpha \otimes e_{p+1}\Big) = -z_{p+1}^l \cdot (r^2)^k \cdot \sum_{\alpha,\beta=1} z_\alpha \otimes \omega_\alpha \wedge \omega_\beta \otimes e_\beta$

Theorem 5.60. *We have*

$$H^1\big(\mathfrak{g}, K; \mathcal{F} \otimes V\big) = \bigoplus_{k=0}^{\infty} \langle\, \big[(r^2)^k \cdot \varphi_{1,1}\big]\rangle.$$

Proof. See Appendix A. □

We now describe the structure of $H^1(\mathfrak{g}, K; \mathcal{F} \otimes V\big)$ as a $\mathfrak{sl}(2,\mathbb{C})-$module.

Theorem 5.61. $\bigoplus_{k=0}^{\infty} \langle\big[(r^2)^k \cdot \varphi_{1,1}\big]\rangle$ *has* $\mathfrak{sl}(2)-$*module structure* $\big([\circ\big)$, *that is,* $\bigoplus_{k=0}^{\infty} \langle\big[(r^2)^k \cdot \varphi_{1,1}\big]\rangle$ *is a lowest weight* $\mathfrak{sl}(2)-$*module. Moreover, it has lowest weight* $\dfrac{(p+3)i}{2}$ *with lowest weight vector* $\big[\varphi_{1,1}\big]$. [5]

Proof. We first check that $\big[\varphi_{1,1}\big]$ is indeed a lowest weight vector with weight $\dfrac{(p+3)i}{2}$.

$$\begin{aligned}
\Big[L \cdot \varphi_{1,1}\Big] &= \Bigg[\Big(-\frac{1}{2}\sum_{\alpha=1}^{p} \frac{\partial^2}{\partial z_\alpha^2} + \frac{1}{2}z_{p+1}^2\Big) \cdot \Big(\sum_{\alpha,\beta=1}^{p} z_\alpha z_\beta \otimes \omega_\alpha \otimes e_\beta\Big)\Bigg] \\
&= \Bigg[-\frac{1}{2}\sum_{\alpha=1}^{p} 2 \otimes \omega_\alpha \otimes e_\alpha + \frac{1}{2}\sum_{\alpha,\beta} z_{p+1}^2 z_\alpha z_\beta \otimes \omega_\alpha \otimes e_\beta\Bigg] \\
&= \Bigg[d\Big(\frac{1}{2}z_{p+1}\sum_{\alpha=1}^{p} z_\alpha \otimes 1 \otimes e_\alpha - \frac{1}{2}(1 \otimes 1 \otimes e_{p+1})\Big)\Bigg] = \big[0\big].
\end{aligned}$$

Next, we check the weight of $\big[\varphi_{1,1}\big]$. We have:

$$\begin{aligned}
\Big[H \cdot \varphi_{1,1}\Big] &= \Bigg[i \sum_{\alpha,\beta,\gamma=1}^{p} z_\gamma \frac{\partial}{\partial z_\gamma}\Big(z_\alpha z_\beta\Big) \otimes \omega_\alpha \otimes e_\beta + \frac{(p-1)i}{2}\varphi_{1,1}\Bigg] \\
&= \Bigg[2i \cdot \varphi_{1,1} + \frac{(p-1)i}{2} \cdot \varphi_{1,1}\Bigg] = \frac{(p+3)i}{2} \cdot \big[\varphi_{1,1}\big].
\end{aligned}$$

Now, one easily sees that

$$\Big[R \cdot (r^2)^{k-1} \cdot \varphi_{1,1}\Big] = \frac{1}{2}\Big[(r^2)^k \cdot \varphi_{1,1}\Big].$$

[5]The Funke-Millson class $[\varphi_{1,1}]$ is therefore a *holomorphic* class, see Remark 5.33.

All that remains is to check what happens when L is applied to $\Big[(r^2)^k \cdot \varphi_{1,1}\Big]$ for $k > 0$.

$$L \cdot \Big[(r^2)^k \cdot \varphi_{1,1}\Big] = \Bigg[-\frac{1}{2} \sum_{\alpha,\beta,\gamma=1}^{p} \frac{\partial^2}{\partial z_\gamma^2}\Big((r^2)^k \cdot z_\alpha z_\beta\Big) \otimes \omega_\alpha \otimes e_\beta + \frac{1}{2} \cdot z_{p+1}^2 \cdot (r^2)^k \cdot \varphi_{1,1}\Bigg]$$
$$= \Bigg[-\Big(2k(k-1) + kp + 4k\Big) \cdot \varphi_{1,1} - (r^2)^k \cdot \sum_{\alpha=1}^{p} 1 \otimes \omega_\alpha \otimes e_\alpha + \frac{1}{2} \cdot z_{p+1}^2 \cdot (r^2)^k \cdot \varphi_{1,1}\Bigg].$$

Now, we have:

$$\mathrm{d}\Bigg(\frac{1}{2} \cdot z_{p+1} \cdot (r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes 1 \otimes e_\alpha - \frac{1}{2} \cdot (r^2)^k \otimes 1 \otimes e_{p+1}\Bigg)$$
$$= -k \cdot (r^2)^{k-1} \cdot \varphi_{1,1} - (r^2)^k \cdot \sum_{\alpha=1}^{p} 1 \otimes \omega_\alpha \otimes e_\alpha + \frac{1}{2} \cdot z_{p+1}^2 \cdot (r^2)^k \cdot \varphi_{1,1}$$
$$\Longrightarrow \Bigg[-(r^2)^k \cdot \sum_{\alpha=1}^{p} 1 \otimes \omega_\alpha \otimes e_\alpha + \frac{1}{2} \cdot z_{p+1}^2 \cdot (r^2)^k \cdot \varphi_{1,1}\Bigg] = \Big[k \cdot (r^2)^{k-1} \cdot \varphi_{1,1}\Big].$$

We use this cohomology relation to obtain that:

$$L \cdot \Big[(r^2)^k \cdot \varphi_{1,1}\Big] = \Bigg(-\Big(2k(k-1) + kp + 4k\Big) + k\Bigg) \cdot \Big[(r^2)^{k-1} \cdot \varphi_{1,1}\Big]$$
$$= -k\Big(2k + (p+1)\Big) \cdot \Big[(r^2)^{k-1} \cdot \varphi_{1,1}\Big].$$

We see that $2k + (p+1) > 0$ for all $k > 0$ and so $L \cdot \Big[(r^2)^k \cdot \varphi_{1,1}\Big] \neq [0]$ for all $k > 0$. $\square$

Computing $\mathbf{H^2}\big(\mathfrak{g}, \mathbf{K}; \mathcal{F} \otimes \mathbf{V}\big)$

Recall that:

$$C_1^2 = \Big[\mathcal{F} \otimes \bigwedge^2 \mathfrak{p}^* \otimes V\Big]^K = \bigoplus_{l,k=0}^{\infty} \big\langle z_{p+1}^l \cdot (r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha \otimes \omega_\alpha \wedge \omega_\beta \otimes e_\beta \big\rangle.$$

One finds that $d\Big(z_{p+1}^l \cdot (r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha \otimes w_\alpha \wedge w_\beta \otimes e_\beta\Big) = 0$ for all $l, k \geq 0$. However,

$$d\Big(\underbrace{-z_{p+1}^l \cdot (r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha \otimes \omega_\alpha \otimes e_{p+1}}_{\in C_1^1}\Big) = z_{p+1}^l \cdot (r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha \otimes w_\alpha \wedge w_\beta \otimes e_\beta,$$

and so $\mathrm{im}(d) = C_1^2$. Therefore we obtain:

Theorem 5.62.

$$H^2(\mathfrak{g}, K; \mathcal{F} \otimes V) = \{0\}.$$

Remark 5.63. Recall that the differential is a map $d : C^i \to C^{i+1}$. As we have found that $C^k = \{0\}$ for all $3 \le k \le p-3$ we have $C^3 = \{0\}$ for $p \ge 6$. Thus, it is no surprise that

$$d\Big(z_{p+1}^l \cdot (r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha \otimes w_\alpha \wedge w_\beta \otimes e_\beta\Big) = 0.$$

The case $p = 5$ can be readily checked and also gives 0, as we would expect.

Computing $\mathbf{H^{p-2}}\big(\mathfrak{g}, \mathbf{K}; \mathcal{F} \otimes \mathbf{V}\big)$

Recall that

$$C_1^{p-2} = \Big[\mathcal{F} \otimes \bigwedge\nolimits^{p-2} \mathfrak{p}^* \otimes V\Big]^K = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l \cdot (r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha \otimes \star(\omega_\alpha \wedge \omega_\beta) \otimes e_\beta \,\rangle.$$

The differential of these $\mathfrak{k}-$invariant vectors is:

$$\begin{aligned}
&d\Big(z_{p+1}^l \cdot (r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha \otimes \star(w_\alpha \wedge w_\beta) \otimes e_\beta\Big) \\
&= -\, 2lk \cdot z_{p+1}^{l-1} \cdot (r^2)^{k-1} \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha^2 \otimes \omega_\alpha \wedge \star(\omega_\alpha \wedge \omega_\beta) \otimes e_\beta \\
&\quad - l \cdot z_{p+1}^{l-1} \cdot (r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} 1 \otimes \omega_\alpha \wedge \star(\omega_\alpha \wedge \omega_\beta) \otimes e_\beta \\
&\quad - 2lk \cdot z_{p+1}^{l-1} \cdot (r^2)^{k-1} \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha z_\beta \otimes \omega_\beta \wedge \star(\omega_\alpha \wedge \omega_\beta) \otimes e_\beta \\
&\quad + z_{p+1}^{l+1} \cdot (r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha^2 \otimes \omega_\alpha \wedge \star(\omega_\alpha \wedge \omega_\beta) \otimes e_\beta \\
&\quad + z_{p+1}^{l+1} \cdot (r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha z_\beta \otimes \omega_\beta \wedge \star(\omega_\alpha \wedge \omega_\beta) \otimes e_\beta \\
&\quad + z_{p+1}^{l}(r^2)^k \sum_{\alpha,\beta=1}^{p} z_\alpha \otimes \omega_\beta \wedge \star(\omega_\alpha \wedge \omega_\beta) \otimes e_{p+1}.
\end{aligned}$$

Observe that the last term in this calculation is of the form $\sum \cdots \otimes \cdots \otimes e_{p+1}$. As no other term in the calculation is of this form we conclude that no combination of the form

$$\sum_i \left(z_{p+1}^{l_i} \cdot (r^2)^{k_i} \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha \otimes \star(w_\alpha \wedge w_\beta) \otimes e_\beta\right)$$

where l_i, $k_i \in \mathbb{Z}_{\ge 0}$ will be in the kernel of d, hence we obtain the following result.

Theorem 5.64.

$$H^{p-2}\big(\mathfrak{g}, K; \mathcal{F} \otimes V\big) = \{0\}$$

Computing $\mathbf{H^{p-1}}\big(\mathfrak{g},\mathbf{K};\mathcal{F}\otimes\mathbf{V}\big)$

Recall that:

$$C_1^{p-1} = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l\cdot(r^2)^k\cdot\star(f)\rangle \bigoplus \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l\cdot(r^2)^k\cdot\star(\varphi_{1,1})\rangle \bigoplus \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l\cdot(r^2)^k\cdot\star(\varphi_{1,0})\rangle$$

where

$$\star(f) = \sum_{\alpha=1}^{p} 1\otimes\star(\omega_\alpha)\otimes e_\alpha,\;\; \star(\varphi_{1,1}) = \sum_{\alpha,\beta=1}^{p} z_\alpha z_\beta\otimes\star(\omega_\alpha)\otimes e_\beta, \star(\varphi_{1,0}) = \sum_{\alpha=1}^{p} z_\alpha\otimes\star(\omega_\alpha)\otimes e_{p+1}.$$

Let $\Omega := \omega_1\wedge ...\wedge\omega_p$. The differentials of these vectors are as follows:

1. $d\Big(z_{p+1}^l\cdot(r^2)^k\cdot\star(f)\Big)$
$$= -2lk\cdot z_{p+1}^{l-1}\cdot(r^2)^{k-1}\cdot\sum_{\alpha=1}^{p} z_\alpha\otimes\Omega\otimes e_\alpha \;+\; z_{p+1}^{l+1}\cdot(r^2)^k\cdot\sum_{\alpha=1}^{p} z_\alpha\otimes\Omega\otimes e_\alpha$$
$$+\, p\cdot z_{p+1}^l\cdot(r^2)^k\cdot\otimes\Omega\otimes e_{p+1}$$

2. $d\Big(z_{p+1}^l\cdot(r^2)^k\cdot\star(\varphi_{1,1})\Big)$
$$= -l(2k+p+1)\cdot z_{p+1}^{l-1}\cdot(r^2)^k\cdot\sum_{\alpha=1}^{p} z_\alpha\otimes\Omega\otimes e_\alpha \;+\; z_{p+1}^{l+1}\cdot(r^2)^{k+1}\cdot\sum_{\alpha=1}^{p} z_\alpha\otimes\Omega\otimes e_\alpha$$
$$+\, z_{p+1}^l\cdot(r^2)^{k+1}\cdot(1\otimes\Omega\otimes e_{p+1})$$

3. $d\Big(z_{p+1}^l\cdot(r^2)^k\cdot\star(\varphi_{1,0})\Big)$
$$= -l(2k+p)\cdot z_{p+1}^{l-1}\cdot(r^2)^k\cdot(1\otimes\Omega\otimes e_{p+1}) \;+\; z_{p+1}^{l+1}\cdot(r^2)^{k+1}\cdot(1\otimes\Omega\otimes e_{p+1})$$
$$+\, z_{p+1}^l\cdot(r^2)^k\cdot\sum_{\alpha=1}^{p} z_\alpha\otimes\Omega\otimes e_\alpha$$

Now, as one can see from the proof of Theorem 5.60 $\big($which give the result for $H_1^1\big)$, finding the kernel of the differential can be very cumbersome using elementary methods. Therefore, we only state a guess for H_1^{p-1}.

Guess 5.65.

$$H_1^{p-1} = H^{p-1}\big(\mathfrak{g},K;\mathcal{F}\otimes V\big) = \{0\}$$

Computing $\mathbf{H^{p}}\big(\mathfrak{g},\mathbf{K};\mathcal{F}\otimes\mathbf{V}\big)$

Recall that

$$C_1^p = \big[\mathcal{F}\otimes 1\otimes V\big]^K = \bigoplus_{l,k=0}^{\infty} \langle z_{p+1}^l\cdot(r^2)^k\cdot\star(\varphi_{0,1})\rangle \bigoplus \bigoplus_{l,k=0}^{\infty} \big\langle z_{p+1}^l\cdot(r^2)^k\cdot 1\otimes\Omega\otimes e_{p+1}\big\rangle$$

where $\Omega := \omega_1 \wedge \omega_2 \wedge ... \wedge \omega_p$ and $\star(\varphi_{0,1}) = \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha$. As $\dim(\mathfrak{p}) = p$, we have that

$$C_1^{p+i} := \Big[\mathcal{F} \otimes \bigwedge^{p+i} \mathfrak{p}^* \otimes V\Big]^K = \big[\mathcal{F} \otimes \{0\} \otimes V\big]^K = \{0\}$$

for all $i \geq 1$. In particular, all $\mathfrak{k}-$invariant vectors in C_1^p will be in the kernel of d since $C_1^{p+1} = \{0\}$. Therefore, we only need to determine the cohomology relations given by $\mathrm{im}\big(d : C_1^{p-1} \to C_1^p\big)$ in order to describe the cohomology group H_1^p. We obtain the following relations.

1. Relation 1.

   (a) For $l, k \geq 1$ we have

$$d\Big(z_{p+1}^l \cdot (r^2)^k \cdot \star(f) - z_{p+1}^l \cdot (r^2)^{k-1} \cdot \star(\varphi_{1,1})\Big)$$
$$= l(p-1) \cdot z_{p+1}^{l-1} \cdot (r^2)^{k-1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha + (p-1) \cdot z_{p+1}^l \cdot (r^2)^k \cdot \big(1 \otimes \Omega \otimes e_{p+1}\big)$$
$$\implies \left[l \cdot z_{p+1}^{l-1} \cdot (r^2)^{k-1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha\right] = \Big[- z_{p+1}^l \cdot (r^2)^k \cdot (1 \otimes \Omega \otimes e_{p+1})\Big]$$

   (b) For $k = 0, l \geq 0$ we have

$$d\Big(z_{p+1}^l \star (f)\Big) = z_{p+1}^{l+1} \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha + p \cdot z_{p+1}^l (1 \otimes \Omega \otimes e_{p+1})$$
$$\implies \Big[p \cdot z_{p+1}^l \cdot (1 \otimes \Omega \otimes e_{p+1})\Big] = \left[-z_{p+1}^{l+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha\right]$$

   (c) Similarly, for $l = 0$, $k \geq 0$, by consideration of $d\Big((r^2)^k \star (f)\Big)$, we have

$$\Big[p \cdot (r^2)^k \cdot (1 \otimes \Omega \otimes e_{p+1})\Big] = \left[-z_{p+1} \cdot (r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha\right]$$

   Observe that with this relation we can completely replace terms of the form $1 \otimes \Omega \otimes e_{p+1}$ with terms of the form $\sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha$ in the cohomology group $H_1^p = H^p(\mathfrak{g}, K; \mathcal{F} \otimes V)$.

2. Relation 2. We now obtain a relation involving only $\sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha$ terms. For all $l, k \geq 0$ we have:

$$d\Big(z_{p+1}^{l} \cdot (r^2)^{k+1} \cdot \star(f) - p z_{p+1}^{l} \cdot (r^2)^{k} \cdot \star(\varphi_{1,1})\Big)$$
$$= l\big(2k(p-1) + p^2 + p - 2\big) \cdot z_{p+1}^{l-1} \cdot (r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha$$
$$- (p-1) \cdot z_{p+1}^{l+1} \cdot (r^2)^{k+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha$$
$$\implies l(2k+p+2)\left[z_{p+1}^{l-1} \cdot (r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha\right] = \left[z_{p+1}^{l+1} \cdot (r^2)^{k+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha\right].$$

Using relation 2 we can reduce the powers of z_{p+1} and r^2. If we repeatedly use the relation until it is not possible to do so anymore, then we will be in one of three situations.

1. We will be left with
$$(r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha$$
for some $k \geq 0$. This is not in the image of $d : C_1^{p-1} \to C_1^{p}$.

2. We will be left with
$$z_{p+1} \cdot (r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha$$
for some $k > 0$. Loosely speaking one may apply the relation again, as even though we seemingly obtain a term containing a negative power of z_{p+1}, the whole term is multiplied by $l = 0$ and so it vanishes. More concretely, one can see that
$$d\Big((r^2)^k \star (f) - p(r^2)^{k-1} \star (\varphi_{1,1})\Big) = -(p-1) \cdot z_{p+1} \cdot (r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha.$$

3. Finally, we could be left with
$$z_{p+1}^{l} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha$$
for some $l \geq 0$. This is not in the image of $d : C^{p-1} \to C^{p}$

These relations give us the following characterisations of H_1^p.

Theorem 5.66.

$$H^p(\mathfrak{g}, K; \mathcal{F} \otimes V) = \bigoplus_{l,k=0}^{\infty} \left\langle \left[z_{p+1}^l \cdot (r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \right] \right\rangle$$

with relation

$$l(2k+p+2) \left[z_{p+1}^{l-1} \cdot (r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \right] = \left[z_{p+1}^{l+1} \cdot (r^2)^{k+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \right].$$

for all $l, k \geq 0$. Equivalently,

$$H^p(\mathfrak{g}, K; \mathcal{F} \otimes V) = \bigoplus_{l=0}^{\infty} \left\langle \left[z_{p+1}^l \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \right] \right\rangle \bigoplus \bigoplus_{k=1}^{\infty} \left\langle \left[(r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \right] \right\rangle$$

We now discuss the $\mathfrak{sl}(2)-$module structure of H_1^p. We will use the latter characterisation of H_1^p from the above theorem as it will be easier to describe the module structure. We have:

Theorem 5.67. $\bigoplus_{l=0}^{\infty} \left\langle \left[z_{p+1}^{2l+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \right] \right\rangle$ *has $\mathfrak{sl}(2)-$module structure*

1. $(\circ] \circ])$ *if p is odd.*
2. $(\circ])$ *if p is even.*

Moreover, $\left[z_{p+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \right]$ *is a highest weight vector with weight* $\frac{(p-1)i}{2}$*. If p is even then this is the only highest weight vector. If p is odd then* $\left[z_{p+1}^{p+2} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \right]$ *is also a highest weight vector.*

Proof. We first check that $\left[z_{p+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \right]$ is a highest weight vector with weight $\frac{(p-1)i}{2}$. We have

$$\begin{aligned}
R \cdot \left[z_{p+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \right] &= \left[R \cdot z_{p+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \right] \\
&= \left[\left(\frac{1}{2} \sum_{\alpha=1}^{p} z_\alpha^2 - \frac{1}{2} \frac{\partial^2}{\partial z_{p+1}^2} \right) \cdot z_{p+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \right] \\
&= \left[\frac{1}{2} \cdot r^2 \cdot z_{p+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \right] \\
&= [0], \text{ using relation 2.}
\end{aligned}$$

Thus $\left[z_{p+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha\right]$ is a highest weight vector. Moreover,

$$\begin{aligned}
H \cdot \left[z_{p+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha\right] &= \left[H \cdot z_{p+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha\right] \\
&= \left[\left(i\Big(\sum_{\alpha=1}^{p} z_\alpha \frac{\partial}{\partial z_\alpha} - z_{p+1}\frac{\partial}{\partial z_{p+1}}\Big) + \frac{(p-1)i}{2}\right) \cdot z_{p+1} \cdot \sum_{\beta=1}^{p} z_\beta \otimes \Omega \otimes e_\beta\right] \\
&= \left[\left(i \cdot z_{p+1} \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha \frac{\partial}{\partial z_\alpha} z_\beta \otimes \Omega \otimes e_\beta + \frac{(p-3)i}{2} \cdot z_{p+1} \cdot \sum_{\beta=1}^{p} z_\beta \otimes \Omega \otimes e_\beta\right.\right] \\
&= \left[\frac{(p-1)i}{2} \cdot z_{p+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha\right] = \frac{(p-1)i}{2}\left[z_{p+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha\right]
\end{aligned}$$

Next, one easily sees that

$$L \cdot \left[z_{p+1}^{2l+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha\right] = \frac{1}{2}\left[z_{p+1}^{2(l+1)+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha\right].$$

Finally, we check what happens when R is applied to $\left[z_{p+1}^{2l+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha\right]$.

$$\begin{aligned}
&R \cdot \left[z_{p+1}^{2l+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha\right] \\
&= \left[\underbrace{\frac{1}{2} \cdot r^2 \cdot z_{p+1}^{2l+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha}_{\text{Use cohomology relation}} - \frac{1}{2} \cdot \frac{\partial^2}{\partial z_{p+1}^2}\Big(z_{p+1}^{2l+1}\Big) \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha\right] \\
&= l\Big((p+2) - (2l+1)\Big) \cdot \left[z_{p+1}^{2l-1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha\right]
\end{aligned}$$

Thus, $R \cdot \left[z_{p+1}^{2l+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha\right] = \big[0\big] \iff 2l+1 = p+2$. That is, if p is odd then

$$R \cdot \left[z_{p+1}^{p+2} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha\right] = \big[0\big]$$

□

Theorem 5.68. $\bigoplus_{l=0}^{\infty} \langle \Big[z_{p+1}^{2l} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \Big] \rangle \bigoplus \bigoplus_{k=1}^{\infty} \langle \Big[(r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \Big] \rangle$ *has* $\mathfrak{sl}(2)-$*module structure:*

1. $\big(\circ \big)$ *if* p *is odd*
2. $\big(\circ \big] \circ \big)$ *if* p *is even.*

Moreover, if p *is even then* $\Big[z_{p+1}^{p+2} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \Big]$ *is the highest weight vector, that is,*

$$R \cdot \Big[z_{p+1}^{p+2} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \Big] = \big[0\big].$$

Proof. Firstly, we easily see that

$$R \cdot \Big[(r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \Big] = \frac{1}{2} \Big[(r^2)^{k+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \Big]$$

and

$$L \cdot \Big[z_{p+1}^{2l} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \Big] = \frac{1}{2} \Big[z_{p+1}^{2(l+1)} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \Big].$$

Thus, we only need to check what happens when we apply L to $\Big[(r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \Big]$ and when we apply R to $\Big[z_{p+1}^{2l} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \Big]$ for $l, k \geq 1$. We have

$$\begin{aligned}
& L \cdot \Big[(r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \Big] \\
&= \Big[-\frac{1}{2} \cdot \sum_{\alpha,\beta=1}^{p} \frac{\partial^2}{\partial z_\beta^2} \Big((r^2)^k \cdot z_\alpha \Big) \otimes \Omega \otimes e_\alpha + \underbrace{\frac{1}{2} \cdot z_{p+1}^2 \cdot (r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha}_{\text{Use cohomology relation}} \Big] \\
&= -\frac{1}{2} \Big(2k-1 \Big) \Big(2k+p \Big) \cdot \Big[(r^2)^{k-1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \Big].
\end{aligned}$$

This has no solutions for $k \in \mathbb{Z}_{\geq 1}$. Finally we have:

$$R \cdot \Big[z_{p+1}^{2l} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \Big]$$

$$= \Bigg[\underbrace{\frac{1}{2} \cdot r^2 \cdot z_{p+1}^{2l} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha}_{\text{Use cohomology relation}} - \frac{1}{2} \cdot \frac{\partial^2}{\partial z_{p+1}^2} \Big(z_{p+1}^{2l} \Big) \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \Bigg]$$

$$= \frac{1}{2}\Big(2l-1\Big)\Big((p+2)-2l\Big) \cdot \Bigg[z_{p+1}^{2l-2} \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \Bigg].$$

Thus, $R \cdot \Bigg[z_{p+1}^{2l} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \Bigg] = \Big[0\Big] \iff 2l = p+2$, that is, when p is even we have

$$R \cdot \Bigg[z_{p+1}^{p+2} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \Omega \otimes e_\alpha \Bigg] = \Big[0\Big].$$

□

5.3 $l > 1$

As we seen in the proof of Theorem 5.60 and by the lack of a result for H_1^{p-1}, determining the kernel of the differential is tricky, even for very small values of l. As one can imagine, for $l > 1$ these computations get much more complicated and are very inefficient. Thus for general $l \in \mathbb{Z}_{\geq 0}$, more sophisticated algebraic machinery is required. However it is not all bad news. In particular, we have a general result for the spaces $C_l^i := C^i\big(\mathfrak{g}, K; \mathcal{F} \otimes \mathrm{Sym}^l(V)\big)$ in the case $3 \leq i \leq p-3$.

Theorem 5.69. *Let $l \in \mathbb{Z}_{\geq 0}$, $3 \leq i \leq p-3$, then*

$$C_l^i = \Big[\mathcal{F} \otimes \bigwedge^i \mathfrak{p}^* \otimes \mathrm{Sym}^l(V) \Big]^K = \{0\}.$$

Proof. We prove the result for $3 \leq i \leq \frac{p}{2}$. The $\mathfrak{k}-$isomorphism, Hodge star, gives the result for $\frac{p}{2} < i \leq p-3$. We proceed with induction. Firstly, the result holds in the cases $l = 0, 1$ as we have seen earlier in the section. We assume the result holds for all $n \in \mathbb{Z}_{>0}$ such that $n < l$. Now recall that

$$\mathrm{Sym}^n(V_+) \otimes \mathrm{Sym}^l(V_+) = \bigoplus_{j=0}^{\lfloor \frac{n}{2} \rfloor} \Bigg(\bigoplus_{k \geq \frac{l-(n-2j)}{2}}^{k \leq \frac{l}{2}} \bigoplus_{0 \leq \zeta \leq l-2k} \bigoplus_{\alpha \in Y(n-2j, l-2k, \zeta)} \Gamma_\alpha$$
$$\oplus \bigoplus_{k=0}^{k < \frac{l-(n-2j)}{2}} \bigoplus_{0 \leq \zeta \leq n-2j} \bigoplus_{\alpha \in Y(l-2k, n-2j, \zeta)} \Gamma_\alpha \Bigg).$$

Observe that for any $l \in \mathbb{Z}_{\geq 0}$, the decomposition contains irreducible representations with associated partitions λ which have positive terms in entries λ_1 and λ_2 only. In particular, the decomposition does not contain any copies of $\Gamma_{\underbrace{(1,...,1)}_{i \text{ entries}}}$, nor does it contain any copies of the representation $\Gamma_{\underbrace{(1,...,1,-1)}_{i \text{ entires}}}$ $\left(\text{the latter is needed for the case } i = \frac{p}{2}\right)$. From this we deduce that for all $n \geq 0$ and all $3 \leq i \leq \frac{p}{2}$ we have

$$\begin{aligned}\dim\left(C_l^i|_{n+}\right) &= \dim\left(\left[\mathrm{Sym}^n(V_+) \otimes \bigwedge^i \mathfrak{p}^* \otimes \mathrm{Sym}^l(V_+)\right]^{SO(p)}\right) \\ &= \dim\left(\mathrm{Hom}_{\mathfrak{so}(p)}\left(\bigwedge^i \mathfrak{p},\ \mathrm{Sym}^n(V_+) \otimes \mathrm{Sym}^l(V_+)\right)\right) = 0\end{aligned}$$

which implies that $C_1^i|_+ = \{0\}$. We may re-introduce z_{p+1} exactly as we did in the cases $l = 0$ and $l = 1$ to obtain $\left[\mathcal{F} \otimes \bigwedge^i \mathfrak{p}^* \otimes \mathrm{Sym}^l(V_+)\right]^K = \{0\}$. Now observe that for any

$$\sum_k p_k \otimes \Omega_k \otimes e_{k_1}e_{k_2}...e_{k_l} \in \mathcal{F} \otimes \bigwedge^i \mathfrak{p}^* \otimes \mathrm{Sym}^l(V)$$

we may express it as

$$\underbrace{\sum_k p_k \otimes \Omega_k \otimes e^+_{k_1}e^+_{k_2}...e^+_{k_l}}_{\in \mathcal{F} \otimes \bigwedge^i \mathfrak{p}^* \otimes \mathrm{Sym}^l(V_+)} + \sum_k p_k \otimes \Omega_k \otimes e_{k_1}e_{k_2}...e_{k_{l-1}}e_{p+1}.$$

Note that the e_{k_j} in the second summand are allowed to be e_{p+1}. By Lemma 5.42, if $\sum_k p_k \otimes \Omega_k \otimes e_{k_1}e_{k_2}...e_{k_l}$ is $\mathfrak{k}-$invariant, then both summands in the above expression must be independently $\mathfrak{k}-$invariant. Observe that:

$$\begin{aligned}&\sum_k p_k \otimes \Omega_k \otimes e_{k_1}e_{k_2}...e_{k_{l-1}}e_{p+1} \in \left[\mathcal{F} \otimes \bigwedge^i \mathfrak{p}^* \otimes \mathrm{Sym}^l(V)\right]^K \\ \iff &\sum_k p_k \otimes \Omega_k \otimes e_{k_1}e_{k_2}...e_{k_{l-1}} \in \left[\mathcal{F} \otimes \bigwedge^i \mathfrak{p}^* \otimes \mathrm{Sym}^{l-1}(V)\right]^K.\end{aligned}$$

Now, by our induction step, $\left[\mathcal{F} \otimes \bigwedge^i \mathfrak{p}^* \otimes \mathrm{Sym}^{l-1}(V)\right]^K = \{0\}$. This result along with the fact that

$$\left[\mathcal{F} \otimes \bigwedge^i \mathfrak{p}^* \otimes \mathrm{Sym}^l(V_+)\right]^K = \{0\}$$

implies that if

$$\sum_k p_k \otimes \Omega_k \otimes e_{k_1}e_{k_2}...e_{k_l} \in \left[\mathcal{F} \otimes \bigwedge^i \mathfrak{p}^* \otimes \mathrm{Sym}^l(V)\right]^K$$

then

$$\sum_k p_k \otimes \Omega_k \otimes e_{k_1}e_{k_2}...e_{k_l} = 0 + 0 = 0.$$

□

Bibliography

[1] Friedrich Wagemann. Introduction to Lie algebra cohomology with a view towards BRST cohomology. (Lecture notes), https://www.math.sciences.univ-nantes.fr/%7e wagemann/LAlecture.pdf, 2019-2020.

[2] A. Borel and N.R. Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups.* 1980.

[3] Jacob Ralston. The relative lie algebra cohomology of the weil representation. *arXiv preprint arXiv:1505.00222*, 2015 (PhD thesis).

[4] Stephen S Kudla and John J Millson. Intersection numbers of cycles on locally symmetric spaces and fourier coefficients of holomorphic modular forms in several complex variables. *Publications Mathématiques de l'IHÉS*, 71:121–172, 1990.

[5] Claude Chevalley and Samuel Eilenberg. Cohomology Theory of Lie Groups and Lie Algebras. *Trans. Am. Math. Soc.*, 63:85–124, 1948.

[6] Jens Funke and John Millson. Cycles with local coefficients for orthogonal groups and vector-valued siegel modular forms. *American Journal of Mathematics*, 128(4):899–948, 2006.

[7] Roger E Howe and Eng Chye Tan. *Non-Abelian Harmonic Analysis: Applications of SL (2,R).* Springer Science & Business Media, 2012.

[8] W. Fulton and J. Harris. Representation theory: A first course. 1991.

[9] Brian Hall. *Lie groups, Lie algebras, and representations: an elementary introduction*, volume 222. Springer, 2015.

[10] S. Lang. *SL2(R).* Graduate texts in mathematics. Springer, 1985.

[11] R. Berndt. *Representations of Linear Groups: An Introduction Based on Examples from Physics and Number Theory.* Vieweg+Teubner Verlag, 2007.

[12] Prof. J. Funke. Lecture notes for module "Representation theory MATH3371", Durham University, 2017-2018.

[13] Dr. J. Shotton. Lecture notes for module "Representation theory MATH3371", Durham University, 2019-2020.

A Proof of Theorem 5.60

The theorem states that:

Theorem.

$$H_1^1 := H^1\big(\mathfrak{g}, K; \mathcal{F} \otimes V\big) = \bigoplus_{k=0}^{\infty} \langle \,\big[(r^2)^k \cdot \varphi_{1,1}\big]\rangle$$

where $\varphi_{1,1} = \displaystyle\sum_{\alpha,\beta=1}^{p} z_\alpha z_\beta \otimes \omega_\alpha \otimes e_\beta$.

Proof. Recall that the differentials of the $\mathfrak{k}-$invariant vectors in C_1^1 are given by:

- $$d\Big(z_{p+1}^l \cdot (r^2)^k \cdot \sum_{\alpha=1}^{p} 1 \otimes \omega_\alpha \otimes e_\alpha\Big) = -\,2lk \cdot z_{p+1}^{l-1} \cdot (r^2)^{k-1} \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha \otimes \omega_\alpha \wedge \omega_\beta \otimes e_\beta$$
$$+\, z_{p+1}^{l+1} \cdot (r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha \otimes \omega_\alpha \wedge \omega_\beta \otimes e_\beta$$
- $$d\Big(z_{p+1}^l \cdot (r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha z_\beta \otimes \omega_\alpha \otimes e_\beta\Big) = l \cdot z_{p+1}^{l-1} \cdot (r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha \otimes \omega_\alpha \wedge \omega_\beta \otimes e_\beta$$
- $$d\Big(z_{p+1}^l \cdot (r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes \omega_\alpha \otimes e_{p+1}\Big) = -z_{p+1}^l \cdot (r^2)^k \cdot \sum_{\alpha,\beta=1} z_\alpha \otimes \omega_\alpha \wedge \omega_\beta \otimes e_\beta.$$

We define the following three polynomials in variables x, y:

a) $p_{a,b}(x, y) := -2abx^{a-1}y^{b-1} + x^{a+1}y^b$

b) $q_{c,d}(x, y) := cx^{c-1}y^d$

c) $r_{f,g}(x, y) := -z^f y^g$

where $a, b, c, d, f, g \in \mathbb{Z}_{\geq 0}$ are fixed. We will adopt the convention that

$$x^n = 0 \text{ if } n < 0$$

and likewise for y^n. Observe that combinations of polynomials of the form p, q, r which sum to 0 is in one-to-one correspondence with combinations of the $\mathfrak{k}-$invariant vectors whose image under the differential is 0. Therefore we will classify all combinations of p, q, r which sum to give 0, and then use this one-to-one correspondence to realise these as combinations of the $\mathfrak{k}-$invariant vectors that are in the kernel of d. We will then show that the only combination of $\mathfrak{k}-$invariant vectors which is not [0] in cohomology are the vectors of the form

$$(r^2)^k \cdot \sum_{\alpha,\beta=1}^{p} z_\alpha z_\beta \otimes \omega_\alpha \otimes e_\beta$$

or scalar multiples thereof, which correspond to the polynomials of the form

$$q_{0,k}(x,y) = 0x^{0-1}y^k = 0.$$

We will now classify all the combinations of polynomials p, q, r which sum to 0 by considering the number of different $p_{a,b}(x,y)$ terms involved. In what follows, we will state the polynomial combinations with at least one of p, q, r having coefficient 1 for simplicity, but of course scalar multiples of all combinations given are also valid.

- 0 $p_{a,b}(x,y)$ terms:

  We have the following combinations which sum to 0 which involve no terms of the form $p_{a,b}(x,y)$:

  1. $q_{0,d}(x,y) = 0x^{0-1}y^d = 0$
  2. $q_{c,d}(x,y) + c \cdot r_{c-1,d}(x,y)$ for $c \geq 1$

- 1 $p_{a,b}(x,y)$ term:

  We have $p_{a,b}(x,y) = -2abx^{a-1}y^{b-1} + x^{a+1}y^b$. Thus, we need to find combinations of polynomials q and r which cancel out $-2abx^{a-1}y^{b-1}$ and $x^{a+1}y^b$. We find that

  - $2abx^{a-1}y^{b-1} = (2ab - \lambda)\frac{1}{a}q_{a,b-1}(x,y) - \lambda r_{a-1,b-1}(x,y)$ for $a, b \geq 1$, $\lambda \in \mathbb{C}$
    Note: if either a or b are 0 then $-2abx^{a-1}y^{b-1}$ vanishes so we have nothing to cancel out.
  - $-x^{a+1}y^b = -(1-\mu)\frac{1}{a+2}q_{a+2,b}(x,y) + \mu r_{a+1,b}(x,y)$, for $a, b \geq 0$, $\mu \in \mathbb{C}$.

  Altogether, this gives us that the two parameter family (with parameters λ and μ) of polynomial combinations involving one $p_{a,b}(x,y)$ term, with a, b fixed, which sum to 0 is

$$\begin{aligned}
&p_{a,b}(x,y) + \frac{1}{a}(2ab-\lambda)q_{a,b-1}(x,y) - \lambda r_{a-1,b-1}(x,y) - \frac{1}{a+2}(1-\mu)q_{a+2,b}(x,y) + \mu r_{a+1,b}(x,y) \\
&= p_{a,b}(x,y) + 2bq_{a,b-1}(x,y) - \frac{\lambda}{a}\Big(q_{a,b-1}(x,y) + ar_{a-1,b-1}(x,y)\Big) \\
&\quad - \frac{1}{a+2}q_{a+2,b}(x,y) + \frac{\mu}{a+2}\Big(q_{a+2,b}(x,y) + (a+2)r_{a+1,b}(x,y)\Big).
\end{aligned}$$

  Observe that the two parameter family of combinations is of the form

$$p_{a,b}(x,y) + 2bq_{a,b-1}(x,y) - \frac{1}{a+2}q_{a+2,b}(x,y) + \text{ terms from the ‘0 } p_{a,b}(x,y) \text{ terms’ case.}$$

  Therefore, the terms involving λ and μ only contribute to combinations we already know. Thus, the only combination in this case which sums to 0 which doesn't contain any combinations of the form 1 and 2 is

  3. $p_{a,b}(x,y) + 2b \cdot q_{a,b-1}(x,y) - \dfrac{1}{a+2} \cdot q_{a+2,b}(x,y)$

- 2 $p_{a,b}(x,y)$ terms:

  i) If the $p_{a,b}(x,y)$ and $p_{a',b'}(x,y)$ polynomials involved share no common terms, then we end up with 2 disjoint combinations from the '1 $p_{a,b}(x,y)$ term' case. Therefore we obtain no new combinations here.

  ii) Suppose then that $p_{a,b}(x,y)$ and $p_{a',b'}(x,y)$ provide cancellation between them. If all terms all cancelled then $p_{a',b'}(x,y) = -p_{a,b}(x,y)$ and this combination is useless. Therefore, we suppose that $p_{a,b}(x,y)$ and $p_{a',b'}(x,y)$ have one term in common, then we obtain a polynomial of the form:

$$\begin{aligned}
&p_{a,b}(x,y) + \frac{1}{2(a+2)(b+1)} p_{a+2,b+1}(x,y) \\
&= -2abx^{a-1}y^{b-1} + x^{a+1}y^b - x^{a+1}y^b + \frac{1}{2(a+2)(b+1)} x^{a+3}y^b \\
&= -2abx^{a-1}y^{b-1} + \frac{1}{2(a+2)(b+1)} x^{a+3}y^b
\end{aligned}$$

  We now look for combinations of polynomials q and r which cancel the terms

$$-2abx^{a-1}y^{b-1} \quad \text{and} \quad \frac{1}{2(a+2)(b+1)} x^{a+3}y^b.$$

  - $2abx^{a-1}y^{b-1} = \frac{1}{a}(2ab-\lambda)q_{a,b-1}(x,y) - \lambda r_{a-1,b-1}(x,y)$ for $a,b \geq 1$, $\lambda \in \mathbb{C}$.
  - $\displaystyle -\frac{1}{2(a+2)(b+1)} x^{a+3}y^b$
$$= -\frac{1}{2(a+2)(b+1)} \left(\frac{1}{a+4}(1-\zeta) q_{a+4,b+1}(x,y) - \zeta r_{a+3,b+1}(x,y) \right), \quad \zeta \in \mathbb{C}$$

  Thus, we have that

$$\begin{aligned}
&p_{a,b}(x,y) + \frac{1}{2(a+2)(b+1)} p_{a+2,b+1}(x,y) \\
&+ \frac{1}{a}(2ab-\lambda) q_{a,b-1}(x,y) - \lambda r_{a-1,b-1}(x,y) \\
&- \frac{1}{2(a+2)(b+1)} \left(\frac{1}{a+4}(1-\zeta) q_{a+4,b+1}(x,y) - \zeta r_{a+3,b+1}(x,y) \right) = 0
\end{aligned} \tag{$*$}$$

  We now introduce a combination of q and r which cancels $x^{a+1}y^b$.

  - $x^{a+1}y^b = \frac{1}{a+2}(1-\mu) q_{a+2,b}(x,y) - \mu r_{a+1,b}(x,y)$, $\mu \in \mathbb{C}$

  We can now add and subtract this combination of q and r to both sides of the equation labelled $(*)$. This has the net affect of adding 0, therefore we obtain:

$$\begin{aligned}
&p_{a,b}(x,y) + \frac{1}{2(a+2)(b+1)} p_{a+2,b+1}(x,y)\\
&+ \frac{1}{a}(2ab-\lambda) q_{a,b-1}(x,y) - \lambda r_{a-1,b-1}(x,y)\\
&+ \frac{1}{a+2}(1-\mu) q_{a+2,b}(x,y) - \mu r_{a+1,b}(x,y)\\
&- \Big(\frac{1}{a+2}(1-\mu) q_{a+2,b}(x,y) - \mu r_{a+1,b}(x,y)\Big)\\
&- \frac{1}{2(a+2)(b+1)}\Big(\frac{1}{a+4}(1-\zeta) q_{a+4,b+1}(x,y) - \zeta r_{a+3,b+1}(x,y)\Big) = 0.
\end{aligned}$$

We may write this as

$$\begin{aligned}
&p_{a,b}(x,y) + 2b q_{a,b-1}(x,y) - \frac{1}{a+2} q_{a+2,b}(x,y)\\
&- \frac{\lambda}{a}\Big(q_{a,b-1}(x,y) + a r_{a-1,b-1}(x,y)\Big)\\
&+ \frac{\mu}{a+2}\Big(q_{a+2,b}(x,y) + (a+2) r_{a+1,b}(x,y)\Big)\\
&- -\\
&+ \frac{1}{2(a+2)(b+1)}\left(p_{a+2,b+1}(x,y) + 2(b+1) q_{a+2,b}(x,y) - \frac{1}{a+4} q_{a+4,b+1}(x,y)\right)\\
&- \frac{\mu}{a+2}\Big(q_{a+2,b}(x,y) + \big(a+2\big) r_{a+1,b}(x,y)\Big)\\
&+ \frac{\zeta}{2(a+2)(b+1)(a+4)}\Big(q_{a+4,b+1}(x,y) + \big(a+4\big) r_{a+3,b+1}(x,y)\Big)
\end{aligned}$$

We recognise this as sum of scalar combinations of the form 1, 2 and 3, therefore we have obtained no new combinations here.

- For greater than 2 $p_{a,b}(x,y)$ terms we use the same method as in the '2 $p_{a,b}(x,y)$ terms' case to show that we once again simply get a sum of combinations of the form 1, 2 and 3.

Having now found all distinct combinations of polynomials which sum to 0, we use the one-to-one correspondence to realise them as combinations of $\mathfrak{k}-$invariant vectors. Recall that in Section 5, we defined

$$f := \sum_{\alpha=1}^{p} 1 \otimes \omega_\alpha \otimes e_\alpha, \quad \varphi_{1,1} = \sum_{\alpha,\beta=1}^{p} z_\alpha z_\beta \otimes \omega_\alpha \otimes e_\beta, \quad \varphi_{1,0} = \sum_{\alpha=1}^{p} z_\alpha \otimes \omega_\alpha \otimes e_{p+1}.$$

With this notation polynomial combinations 1, 2 and 3 correspond to the following $\mathfrak{k}-$invariant vector combinations:

1. $(r^2)^k \cdot \varphi_{1,1}$

2. $z_{p+1}^l \cdot (r^2)^k \cdot \varphi_{1,1} + l \cdot z_{p+1}^{l-1} \cdot (r^2)^k \cdot \varphi_{1,0}$ for $l \geq 1$. Note that the case $l = 0$ simply gives us vector combination 1.

3. $z_{p+1}^l \cdot (r^2)^k \cdot f + 2k \cdot z_{p+1}^l \cdot (r^2)^{k-1} \cdot \varphi_{1,1} - \dfrac{1}{l+2} \cdot z_{p+1}^{l+2} \cdot (r^2)^k \cdot \varphi_{1,1}$

Recall from our calculations when computing H_1^0 that

A) $d\Big(z_{p+1}^l \cdot (r^2)^k \cdot \displaystyle\sum_{\alpha=1}^{p} z_\alpha \otimes 1 \otimes e_\alpha\Big)$
$= -2lk \cdot z_{p+1}^{l-1} \cdot (r^2)^{k-1} \cdot \varphi_{1,1} \;-\; l \cdot z_{p+1}^{l-1} \cdot (r^2)^k \cdot f \;+\; z_{p+1}^{l+1} \cdot (r^2)^k \cdot \varphi_{1,1} \;+\; z_{p+1}^l \cdot (r^2)^k \cdot \varphi_{1,0}.$

B) $d\Big(z_{p+1}^l \cdot (r^2)^k \cdot \otimes 1 \otimes e_{p+1}\Big) = -2lk \cdot z_{p+1}^{l-1} \cdot (r^2)^{k-1} \cdot \varphi_{1,0} + z_{p+1}^{l+1} \cdot (r^2)^k \cdot \varphi_{1,0} + z_{p+1}^l \cdot (r^2)^k \cdot f.$

Thus, one sees that
$$(r^2)^k \varphi_{1,1} \;\notin \operatorname{im}\big(d : C_1^0 \to C_1^1\big)$$
for all $k \in \mathbb{Z}_{\geq 0}$, so $\Big[(r^2)^k \varphi_{1,1}\Big]$ is a non-zero class in H_1^1. Now, for vector combination 2, that is,
$$z_{p+1}^l \cdot (r^2)^k \cdot \varphi_{1,1} + l \cdot z_{p+1}^{l-1} \cdot (r^2)^k \cdot \varphi_{1,0}$$
where $l \geq 1$, we consider the following vector in the image of d:

$$d\Big(z_{p+1}^l \cdot (r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes 1 \otimes e_\alpha + z_{p+1}^{l-1} \cdot (r^2)^k \cdot \otimes 1 \otimes e_{p+1}\Big)$$
$$= \Big(-2lk \cdot z_{p+1}^{l-1} \cdot (r^2)^{k-1} + z_{p+1}^{l+1} \cdot (r^2)^k\Big)\varphi_{1,1} + \Big((l+1) \cdot z_{p+1}^l \cdot (r^2)^k - 2l(l-1)k \cdot z_{p+1}^{l-2} \cdot (r^2)^{k-1}\Big)\varphi_{1,0}.$$

This gives us the following relation in the cohomology group H_1^1:

$$\Big[z_{p+1}^{l+1} \cdot (r^2)^k \cdot \varphi_{1,1} + (l+1) \cdot z_{p+1}^l \cdot (r^2)^k \cdot \varphi_{1,0}\Big] = 2lk\Big[z_{p+1}^{l-1} \cdot (r^2)^{k-1} \cdot \varphi_{1,1} + (l-1) \cdot z_{p+1}^{l-2} \cdot (r^2)^{k-1} \cdot \varphi_{1,0}\Big]$$

for $l \geq 2, k \geq 1$. Therefore, given any cohomology class of the form
$$\Big[z_{p+1}^{l+1} \cdot (r^2)^k \cdot \varphi_{1,1} + (l+1) \cdot z_{p+1}^l \cdot (r^2)^k \cdot \varphi_{1,0}\Big]$$
we may always reduce it, using the relation, to a class of the form
$$\Big[z_{p+1}^{a+1} \cdot (r^2)^b \cdot \varphi_{1,1} + (a+1) \cdot z_{p+1}^a \cdot (r^2)^b \cdot \varphi_{1,0}\Big]$$
where $a \leq 1$ or $b = 0$ (or both). We now consider the various cases:

- $a = 1$, $b \geq 1$.

  Suppose we have reduced the class to
  $$\Big[z_{p+1}^2 \cdot (r^2)^b \cdot \varphi_{1,1} + 2 \cdot z_{p+1} \cdot (r^2)^b \cdot \varphi_{1,0}\Big].$$
  Then consider the differential
  $$\begin{aligned} &d\Big(z_{p+1} \cdot (r^2)^b \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes 1 \otimes e_\alpha + (r^2)^b \cdot 1 \otimes 1 \otimes e_{p+1}\Big) \\ &= \Big(-2b \cdot (r^2)^{b-1} + z_{p+1}^2 \cdot (r^2)^b\Big)\varphi_{1,1} + \Big(2 \cdot z_{p+1} \cdot (r^2)^b\Big)\varphi_{1,0}. \end{aligned}$$
  This implies that in the cohomology group H_1^1 we have
  $$\Big[z_{p+1}^2 \cdot (r^2)^b \cdot \varphi_{1,1} + 2 \cdot z_{p+1} \cdot (r^2)^b \cdot \varphi_{1,0}\Big] = \Big[2b \cdot (r^2)^{b-1} \cdot \varphi_{1,1}\Big].$$
  This is exactly the cohomology class we have already found.

- $a = 0$, $b \geq 1$

  Now suppose we have reduced to the class
  $$\Big[z_{p+1} \cdot (r^2)^b \cdot \varphi_{1,1} + (r^2)^b \cdot \varphi_{1,0}\Big].$$
  Then, observe that
  $$d\Big((r^2)^b \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes 1 \otimes e_\alpha\Big) = z_{p+1} \cdot (r^2)^b \cdot \varphi_{1,1} + (r^2)^b \cdot \varphi_{1,0}.$$
  Thus in the cohomology group H_1^1 we have
  $$\Big[z_{p+1} \cdot (r^2)^b \cdot \varphi_{1,1} + (r^2)^b \cdot \varphi_{1,0}\Big] = \big[0\big].$$

- $a \geq 2$, $b = 0$.

  If we reduce to the class
  $$\Big[z_{p+1}^{a+1} \cdot \varphi_{1,1} + (a+1) \cdot z_{p+1}^a \cdot \varphi_{1,0}\Big].$$
  Then, observe that
  $$\begin{aligned} &d\Big(z_{p+1}^a \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes 1 \otimes e_\alpha + a \cdot z_{p+1}^{a-1} \cdot 1 \otimes 1 \otimes e_{p+1}\Big) \\ &= \Big(-a \cdot z_{p+1}^{a-1} \cdot f + z_{p+1}^{a+1} \cdot \varphi_{1,1} + z_{p+1}^a \cdot \varphi_{1,0}\Big) + \Big(a \cdot z_{p+1}^a \cdot \varphi_{1,0} + a \cdot z_{p+1}^{a-1} \cdot f\Big) \\ &= z_{p+1}^{a+1} \cdot \varphi_{1,1} + (a+1) \cdot z_{p+1}^a \cdot \varphi_{1,0} \end{aligned}$$
  and so we have
  $$\Big[z_{p+1}^{a+1} \cdot \varphi_{1,1} + (a+1) \cdot z_{p+1}^a \cdot \varphi_{1,0}\Big] = \big[0\big].$$

- $a = 1$, $b = 0$

  Suppose we reduce to the class
$$\Big[z_{p+1}^2 \cdot \varphi_{1,1} + 2 \cdot z_{p+1} \cdot \varphi_{1,0}\Big].$$
  Then, observe that
$$d\bigg(z_{p+1} \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes 1 \otimes e_\alpha \ + \ 1 \otimes 1 \otimes e_{p+1}\bigg) = z_{p+1}^2 \cdot \varphi_{1,1} + 2 \cdot z_{p+1} \cdot \varphi_{1,0}$$
  and so we have
$$\Big[z_{p+1}^2 \cdot \varphi_{1,1} + 2 \cdot z_{p+1} \cdot \varphi_{1,0}\Big] = [0].$$

- $a = 0$, $b = 0$.

  Finally, if we reduce to the class
$$\Big[z_{p+1} \cdot \varphi_{1,1} + \varphi_{1,0}\Big]$$
  then observe that
$$d\bigg(\sum_{\alpha=1}^{p} z_\alpha \otimes 1 \otimes e_\alpha \bigg) = z_{p+1} \cdot \varphi_{1,1} + \varphi_{1,0}$$
  and so in H_1^1 we have
$$\Big[z_{p+1} \cdot \varphi_{1,1} + \varphi_{1,0}\Big] = [0].$$

So we see that vector combination 2 does not give us any additional non-zero cohomology classes. Finally, we consider vector combination 3, that is,
$$z_{p+1}^l \cdot (r^2)^k \cdot f + 2k \cdot z_{p+1}^l \cdot (r^2)^{k-1} \cdot \varphi_{1,1} - \frac{1}{l+2} \cdot z_{p+1}^{l+2} \cdot (r^2)^k \cdot \varphi_{1,1}.$$
Adding a vector which is in the image of d to the above vector will not change the above vector's class in cohomology. Thus, consider the vector
$$d\bigg(\frac{1}{l+1} \cdot z_{p+1}^{l+1} \cdot (r^2)^k \cdot \sum_{\alpha=1}^{p} z_\alpha \otimes 1 \otimes e_\alpha \bigg)$$
$$= -2k \cdot z_{p+1}^l \cdot (r^2)^{k-1} \cdot \varphi_{1,1} \ - \ z_{p+1}^l \cdot (r^2)^k \cdot f \ + \ \frac{1}{l+1} \cdot z_{p+1}^{l+2} \cdot (r^2)^k \cdot \varphi_{1,1} \ + \ \frac{1}{l+1} \cdot z_{p+1}^{l+1} \cdot (r^2)^k \cdot \varphi_{1,0}.$$
Adding this to vector combination 3 gives
$$\frac{1}{(l+1)(l+2)} \bigg(z_{p+1}^{l+2} \cdot (r^2)^k \cdot \varphi_{1,1} + (l+2) \cdot z_{p+1}^{l+1} \cdot (r^2)^k \cdot \varphi_{1,0} \bigg)$$
which is simply a multiple of vector combination 2, thus we obtain no new cohomology classes from vector combination 3. In conclusion, the only non-zero cohomology classes in H_1^1 are of the form
$$\big[(r^2)^k \cdot \varphi_{1,1}\big]$$
where $k \in \mathbb{Z}_{\geq 0}$ or scalar multiples thereof, as required. □