Contramodules for Algebraic Groups.

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There are two main aims of this talk:

1) Briefly introduce contramodules

2) Construct projective covers of simple G-modules (in the category $k[G]$-Contra.)

Note: This is a contra-analog of the "Inverse limit theorem."

Part 1: What are contramodules? A motivating definition.

Let A be an algebra over a field k. Let M be an A-module. Two natural ways to view this are:

$$\begin{aligned} \rho : A \otimes M &\longrightarrow M \\ a \otimes m &\longmapsto a.m \end{aligned}$$

Reversing arrows in this defn (and replacing A algebra with C coalgebra) defines C-comodules.

$$\begin{aligned} \tau : M &\longrightarrow \mathrm{Hom}_k(A, M) \\ m &\longmapsto (a \longmapsto a.m) \end{aligned}$$

Reversing arrows here (with algebra $A \rightsquigarrow$ coalgebra C) defines C-contramodules

Contramodules of the coalgebra $k[G]$.

G algebraic group over $k = \bar{k}$ with co-ord ring $k[G]$.

$m : G \times G \longrightarrow G \quad \rightsquigarrow \quad m_* : k[G] \longrightarrow k[G] \otimes k[G].$

or "Δ"

Both $k[G]$-comodules & $k[G]$-contramodules give us a way to obtain repns of G.

- Comodules : $\quad V \longrightarrow V \otimes k[G]$

Gives G-repn. via $G \times V \longrightarrow G \times V \otimes k[G] \xrightarrow{\text{eval.}} V$

$\{\text{rational } G\text{-modules}\} \longleftrightarrow \{k[G]\text{-comodules}\}$

- Contramodules : $\quad \mathrm{Hom}_k(k[G], V) \longrightarrow V$

$$G \times V \subset k[G]^* \otimes V \quad (k[G]^* \otimes V \subset \mathrm{Hom}_k(k[G], V))$$

This composition is a G-action on V.

An important example of contramodules

- If M is a $k[G]$-comodule, $\rho: M \to M \otimes k[G]$ then $\mathrm{Hom}_k(M, V)$ is a contramodule via:

  V any vector space.

$$\mathrm{Hom}_k(k[G], \mathrm{Hom}_k(M,V)) \xrightarrow{\otimes\dashv\mathrm{Hom}} \mathrm{Hom}_k(M \otimes k[G], V) \xrightarrow{\rho_*} \mathrm{Hom}_k(M,V)$$

$\to$ In particular for $M = k[G]$ (as a comodule over itself.)

<u>Lemma</u>: For any contramodule B we have:

$$\mathrm{Hom}^{k[G]}(\mathrm{Hom}_k(k[G],V), B) \cong \mathrm{Hom}_k(V, B).$$

<u>Consequences</u>: 1. $\mathrm{Hom}_k(k[G], V)$ is a free contramodule.

2. From contra-action $\mathrm{Hom}_k(k[G], B) \twoheadrightarrow B$:
   B projective $\Leftrightarrow$ $B \underset{\oplus}{\subseteq}$ a free contramodule.

Part 2: Constructing projective covers.

- G simply connected, semisimple alg group over $k=\overline{k}$ with $\operatorname{char} k = p$.

Frobenius morphism. "The rth Frobenius kernel."

Have: $F: G \longrightarrow G$, with $G_r := \ker(F^r)$, $r>0$

- λ dominant weight $\rightsquigarrow L(\lambda)$ simple G-module.

  λ p-restricted dom. weight $\rightsquigarrow L(1,\lambda)$ simple G_1-module

- G-structures: If $V \in G_1\text{-Mod}$ and $W \in G\text{Mod}$ s.t. $W|_{G_1} \cong V$ then <u>W is a G-structure on V</u>

Assume $P(\lambda)$ is a G-structure on $P(1,\lambda)$ – the G_1-projective cover of $L(1,\lambda)$.

(– For $p \geq 2h-4$ they exist (Jantzen, C. Bendel, D. Nakano, C. Pillen, P. Sobaje '22)
– Conjectured to always exist (Humphreys & Verma))

Fix λ - dominant. Write $\lambda = \sum_{i=0}^{s} p^i \lambda_i$ $\left(\begin{matrix}\lambda_i \\ p\text{-restricted}\end{matrix}\right)$.

For $m > s$ define:

$$P_{\lambda,m} := P(\lambda_0) \otimes P(\lambda_1)^{Fr} \otimes \dots \otimes P(\lambda_s)^{Fr^s} \otimes P(0)^{Fr^{s+1}} \otimes \dots \otimes P(0)^{Fr^{m-1}}.$$

Fix projection $q: P(0) \to L(0)$
(this induces projections $q_{m+1}: P_{\lambda,m+1} \twoheadrightarrow P_{\lambda,m}$)

Define $P_\lambda := \varprojlim P_{\lambda,m}$ = limit of diagram $\left(P_{\lambda,m} \twoheadleftarrow P_{\lambda,m+1} \twoheadleftarrow P_{\lambda,m+2} \twoheadleftarrow \dots\right)$

$\boxed{\text{Theorem}}$

P_λ is the projective cover of $L(\lambda)$.

Remainder of talk: Idea behind the proof.

- In 1980, Donkin give a "short simple minded proof of the direct limit theorem."

- Try to modify this proof in our "dual case":

⚠ Problem: In Donkin's proof he makes use of the fact that every comodule is a direct limit (a.k.a union) of it's finite dimensional subcomodules.

✗ The analogous statement for contramodules is not true. (c.f. Positselski: Contramodules, §1.5).

This suggests that using $\mathrm{Hom}^{k[G]}(P_\lambda, -)$ to check projectiveness is not the way to go... So?

Need construction known as $\mathrm{Cohom}_{k[G]}$:

- Takes a comodule M, and a contramodule B.
- $\mathrm{Cohom}_{k[G]}(M, B) = \mathrm{Hom}_k(M, B) \big/ \text{stuff}$

Think $M \otimes_R N$ for R-modules

Lemma

Let $V, W \in k[G]\text{-}\mathrm{Comod}_{f.d.}$ then:

$$\mathrm{Cohom}_{k[G]}(V, W) \cong \mathrm{Hom}_{k[G]}(W, V)^*$$

(viewed as a contramodule via $\mathrm{Hom}_k(k[G], W) \cong k[G]^* \otimes W \to k[G]^* \otimes k[G] \otimes W \to W$.

denotes $k[G]$-comodule homomorphisms.

In particular, one can show for a fixed λ:

$$\dim\left(\mathrm{Cohom}_{k[G]}(V, P_{\lambda,m})\right) = f(V), \quad m \gg 0$$

where $f(V)$ = composition multiplicity of $L(\lambda)$ in V.

Q: How is Cohom useful to us?

Answer:

<u>Lemma (Positselski, '15)</u>

A $k[G]$-contramodule P is projective

$\Leftrightarrow$ $\mathrm{Cohom}_{k[G]}(-, P) : k[G]\text{-}\mathrm{Comod}^{op}_{f.d.} \to k\text{-}\mathrm{Vect}$

is exact.

By using $\mathrm{Cohom}_{k[G]}$ we only need to check exactness for finite-dim'l objects!

proof sketch of main Theorem

- Define $\mathrm{Rad}^{k[G]}(P_\lambda) = \varprojlim_m \mathrm{Rad}_{k[G_m]}(P_{\lambda,m})$.
  - This satisfies conditions we want $\rightarrow$ $P_\lambda / \mathrm{Rad}(P_\lambda) \cong L(\lambda)$.

    $\rightarrow$ Superfluous in P_λ
- To see that P_λ projective:

For V f.d. comodule, one can show:

$$\mathrm{Cohom}_{k[G]}(V, P_\lambda) = \varprojlim \mathrm{Cohom}_{k[G]}(V, P_{\lambda,m}).$$

But by lemma: $\dim(\mathrm{Cohom}_{k[G]}(V, P_{\lambda,m})) = f(V)$, $m \gg 0$

$$\Rightarrow \dim(\mathrm{Cohom}_{k[G]}(V, P_\lambda)) = f(V)$$

Finally, f is additive on S.E.Ss of finite dimensional comodules

$\left(\text{i.e } 0 \to U \to V \to W \to 0 \text{ exact } \Rightarrow f(V) = f(U) + f(W) \right)$

we conclude that $\mathrm{Cohom}_{k[G]}(-, P_\lambda)$ is exact

as functor from $k[G]\text{-}\mathrm{Comod}_{f.d}^{op} \longrightarrow k\text{-}\mathrm{Vect}$

$\overset{\text{Positselski}}{\Longrightarrow}$ P_λ projective.

Thank You!